

Subanalytic topologies I. Construction of sheaves

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January 1, 2013

Abstract

On a real analytic manifold M , we construct what we call the linear subanalytic Grothendieck topology M_{sal} together with the natural morphism of sites $\rho: M_{\text{sa}} \rightarrow M_{\text{sal}}$, where M_{sa} is the usual subanalytic site. Our first result is that the derived direct image functor $R\rho_*$ admits a right adjoint, allowing us to associate functorially a sheaf (in the derived sense) on M_{sa} to a presheaf on M_{sa} satisfying suitable properties.

We apply this construction to various presheaves on real manifolds, such as the presheaves of \mathcal{C}^∞ -functions with temperate growth of a given order at the boundary or with Gevrey growth at the boundary. On a complex manifold endowed with the subanalytic topology, the Dolbeault complexes associated with these new sheaves allow us to obtain various filtrations on the sheaf of holomorphic functions.

Applications to holonomic \mathcal{D} -modules will be developed in a forthcoming paper.

Contents

1	Subanalytic topologies	3
2	Γ-acyclic sheaves	14
3	The functor $\rho_{\text{sal}}^!$	18
4	Open sets with Lipschitz boundaries	22

5	Construction of sheaves I	28
6	Construction of sheaves II	34

Introduction

Consider a real analytic manifold M endowed with the subanalytic topology M_{sa} introduced in [KS01] and let us say that two relatively compact subanalytic open subsets U_1 and U_2 of M_{sa} are 1-regularly situated if there is a constant C such that the distance of any $x \in M$ to $M \setminus (U_1 \cup U_2)$ is bounded by C -times the maximum of the distance of x to $M \setminus U_i$ ($i = 1, 2$).

Let \mathbf{k} be a commutative unital Noetherian ring with finite global dimension and let F be a presheaf of \mathbf{k} -modules on M_{sa} with the property that, for any open sets U_1 and U_2 as above, the Mayer-Vietoris sequence $0 \rightarrow F(U_1 \cup U_2) \rightarrow F(U_1) \oplus F(U_2) \rightarrow F(U_1 \cap U_2) \rightarrow 0$ is exact. We shall functorially associate to such a presheaf F an object \tilde{F} of the derived category of sheaves $\mathbf{D}^b(\mathbf{k}_{M_{\text{sa}}})$ such that $\mathbf{R}\Gamma(U; \tilde{F})$ is in degree 0 and is isomorphic to $F(U)$ as soon as the open relatively compact subanalytic subset U of M has a Lipschitz boundary.

For that purpose we introduce the site M_{sal} whose objects are the same as those of M_{sa} , namely the relatively compact subanalytic open subsets, but the coverings are the finite coverings which are 1-regularly situated (see Definition 1.1 for details). There is a natural morphism of sites $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$ and we are reduce to prove the two results (see Theorems 3.11 and 4.11):

- (1) the functor $\mathbf{R}\rho_{\text{sal}*}$ admits a right adjoint,
- (2) if U has a Lipschitz boundary, then the object $\mathbf{R}\rho_{\text{sal}*}\mathbf{k}_U$ is concentrated in degree 0.

We apply these constructions to the case where F is a presheaf of tempered \mathcal{C}^∞ -functions with growth of a given order or a presheaf of \mathcal{C}^∞ -functions with Gevrey growth at the boundary. Sobolev sheaves will be treated by G. Lebeau in a forthcoming paper [Le12].

Then on a complex manifold X , regarding the sheaf \mathcal{O}_X of holomorphic functions as a Dolbeaut complex, we construct a natural *filtration by the order* on the sheaf $\mathcal{O}_X^{\text{temp}}$ of temperate holomorphic functions and a natural *Gevrey filtration* on the sheaf \mathcal{O}_X .

In forthcoming papers we will:

- (i) study the operations on sheaves on the linear subanalytic sites,

- (ii) study the filtration by the order on the sheaf $\mathcal{O}_X^{\text{temp}}$ (using the tools of [Sn99]) and, as a byproduct, endow regular holonomic \mathcal{D} -modules with the filtration by the order,
- (iii) use the Gevrey sheaves in the study of irregular holonomic \mathcal{D} -modules.

Acknowledgments This paper has essentially been written during two stays of the authors at the Research Institute for Mathematical Sciences at Kyoto University in 2011 and 2012 and we wish to thank this institute for its hospitality. During our stays we had, as usual, extremely enlightening discussions with Masaki Kashiwara and we warmly thank him here.

We have also been very much stimulated by the interest of Gilles Lebeau for sheafifying the classical Sobolev spaces and it is a pleasure to thank him here.

Finally this paper would be of little interest without Theorem 4.8 whose proof has been proposed to us by Adam Parusinski, and we are extremely grateful to him.

1 Subanalytic topologies

Usual notations

We shall mainly follow the notations of [KS90, KS01] and [KS06].

In this paper, we denote by \mathbf{k} a commutative unital *Noetherian* ring with finite global dimension. Unless otherwise specified, a manifold means a real analytic manifold.

For a subset A in a topological space X , \overline{A} denotes its closure, $\text{Int } A$ its interior and ∂A its boundary, $\partial A = \overline{A} \setminus \text{Int } A$.

If \mathcal{C} is an additive category, we denote by $\text{C}(\mathcal{C})$ the additive category of complexes in \mathcal{C} . If \mathcal{C} is an abelian category, we denote by $\text{D}(\mathcal{C})$ its (unbounded) derived category. For $* = +, -, b$ we also consider the full additive subcategory $\text{C}^*(\mathcal{C})$ of $\text{C}(\mathcal{C})$ consisting of complexes bounded from below (resp. from above, resp. bounded) and similarly with $\text{D}^*(\mathcal{C})$.

For a site \mathcal{T} , we denote by $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ the abelian category of sheaves of \mathbf{k} -modules on \mathcal{T} . Recall that $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ is a Grothendieck category. We write $\text{D}^*(\mathbf{k}_{\mathcal{T}})$ instead of $\text{D}^*(\text{Mod}(\mathbf{k}_{\mathcal{T}}))$ ($* = \emptyset, +, -, b$).

We will often use the following well-known fact. For any $F \in \text{D}(\mathbf{k}_{\mathcal{T}})$ and any $i \in \mathbb{Z}$, the cohomology sheaf $H^i(F)$ is the sheaf associated with the

presheaf $U \mapsto H^i(U; F)$. In particular, if $H^i(U; F) = 0$ for all $U \in \mathcal{T}$, then $H^i(F) \simeq 0$.

For an object U of \mathcal{T} , recall that there is a sheaf naturally attached to U (see *e.g.* [KS06, § 17.6]). We shall denote it here by $\mathbf{k}_{U\mathcal{T}}$ or simply \mathbf{k}_U if there is no risk of confusion. This is the sheaf associated with the presheaf (see *loc. cit.* Lemma 17.6.11):

$$V \mapsto \oplus_{V \rightarrow U} \mathbf{k}.$$

The functor “associated sheaf” is exact. It follows that, if $V \rightarrow U$ is a monomorphism in \mathcal{T} , then the natural morphism $\mathbf{k}_{V\mathcal{T}} \rightarrow \mathbf{k}_{U\mathcal{T}}$ also is a monomorphism.

For a real analytic manifold M , one denotes by $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the category of \mathbb{R} -constructible sheaves on M . One denotes by $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$ the full triangulated subcategory of $\text{D}^b(\mathbf{k}_M)$ consisting of objects with \mathbb{R} -constructible cohomologies.

The site M_{sa}

We shall mainly use the subanalytic topology introduced in [KS01]. In *loc. cit.*, sheaves on the subanalytic topology are studied in the more general framework of indsheaves. We refer to [Pr08] for a direct and more elementary treatment of subanalytic sheaves.

Let M be a real analytic manifold and denote by $\text{Op}_{M_{\text{sa}}}$ the category of relatively compact subanalytic open subsets of M , the morphisms being the inclusion morphisms. Recall that one endows $\text{Op}_{M_{\text{sa}}}$ with a Grothendieck topology by saying that a family $\{U_i\}_{i \in I}$ of objects $\text{Op}_{M_{\text{sa}}}$ is a covering of $U \in \text{Op}_{M_{\text{sa}}}$ if $U_i \subset U$ for all $i \in I$ and there exists a finite subset $J \subset I$ such that $\bigcup_{j \in J} U_j = U$.

One shall be aware that if U is an open subset of M , we may endow it with the subanalytic topology U_{sa} , but this topology does not coincide in general with the topology induced by M .

We denote by $\rho_{\text{sa}}: M \rightarrow M_{\text{sa}}$ (or simply ρ) the natural morphism of sites. We get the pairs of adjoint functors $(\rho_{\text{sa}}^{-1}, \rho_{\text{sa}*})$ and $(\rho_{\text{sa}}^{-1}, \text{R}\rho_{\text{sa}*})$:

$$(1.1) \quad \text{Mod}(\mathbf{k}_M) \xrightleftharpoons[\rho_{\text{sa}}^{-1}]{\rho_{\text{sa}*}} \text{Mod}(\mathbf{k}_{M_{\text{sa}}}), \quad \text{D}^b(\mathbf{k}_M) \xrightleftharpoons[\rho_{\text{sa}}^{-1}]{\text{R}\rho_{\text{sa}*}} \text{D}^b(\mathbf{k}_{M_{\text{sa}}})$$

The functor ρ_{sa}^{-1} also admits a left adjoint functor $\rho_{\text{sa}!}$. For $F \in \text{Mod}(\mathbf{k}_M)$, $\rho_{\text{sa}!}F$ is the sheaf on M_{sa} associated with the presheaf $U \mapsto F(\overline{U})$.

Recall that $\rho_{\text{sa}*}$ is fully faithful and is exact when restricted to the subcategory $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$. Hence we shall consider this last category both as a full subcategory of $\text{Mod}(\mathbf{k}_M)$ and a full subcategory of $\text{Mod}(\mathbf{k}_{M_{\text{sa}}})$.

For $U \in \text{Op}_{M_{\text{sa}}}$ we have the sheaf $\mathbf{k}_{UM_{\text{sa}}} \simeq \rho_{\text{sa}*} \mathbf{k}_{UM}$ on M_{sa} that we simply denote by \mathbf{k}_U .

The site M_{sal}

Let us choose a distance d on M such that, for any $x \in M$ and any local chart $(U, \varphi: U \hookrightarrow \mathbb{R}^n)$ around x , there exists a neighborhood of x over which d is Lipschitz equivalent to the pull-back of the Euclidean distance by φ .

Definition 1.1. Let $\{U_i\}_{i \in I}$ be a finite family in $\text{Op}_{M_{\text{sa}}}$. We say that this family is 1-regularly situated if there is a constant C such that for any $x \in M$

$$(1.2) \quad d(x, M \setminus \bigcup_{i \in I} U_i) \leq C \cdot \max_{i \in I} d(x, M \setminus U_i).$$

Of course, this definition does not depend on the choice of the distance d .

Example 1.2. On \mathbb{R}^2 with coordinates (x_1, x_2) consider the open sets:

$$\begin{aligned} U_1 &= \{(x_1, x_2); x_2 > -x_1^2, x_1 > 0\}, \\ U_2 &= \{(x_1, x_2); x_2 < x_1^2, x_1 > 0\}, \\ U_3 &= \{(x_1, x_2); x_1 > -x_2^2, x_2 > 0\}. \end{aligned}$$

Then $\{U_1, U_2\}$ is not 1-regularly situated. Indeed, set $W := U_1 \cup U_2 = \{x_1 > 0\}$. Then, if $x = (x_1, 0)$, $x_1 > 0$, $d(x, \mathbb{R}^2 \setminus W) = x_1$ and $d(x, \mathbb{R}^2 \setminus U_i)$ ($i = 1, 2$) is less than x_1^2 .

On the other hand $\{U_1, U_3\}$ is 1-regularly situated. Indeed,

$$d(x, \mathbb{R}^2 \setminus (U_1 \cup U_3)) \leq \sqrt{2} \max(d(x, \mathbb{R}^2 \setminus U_1), d(x, \mathbb{R}^2 \setminus U_3)).$$

Definition 1.3. A linear covering of U is a small family $\{U_i\}_{i \in I}$ of objects of $\text{Op}_{M_{\text{sa}}}$ such that $U_i \subset U$ for all $i \in I$ and

$$(1.3) \quad \begin{aligned} &\text{there exists a finite subset } I_0 \subset I \text{ such that the family } \{U_i\}_{i \in I_0} \\ &\text{is 1-regularly situated and } \bigcup_{i \in I_0} U_i = U. \end{aligned}$$

Let $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ be two families of objects of $\text{Op}_{M_{\text{sa}}}$. Recall that one says that $\{U_i\}_{i \in I}$ is a refinement of $\{V_j\}_{j \in J}$ if for any $i \in I$, there exists $j \in J$ with $U_i \subset V_j$.

Lemma 1.4. *The family of linear coverings satisfies the axioms of Grothendieck topologies below (see [KS06, § 16.1]).*

COV1 $\{U\}$ is a covering of U , for any $U \in \text{Op}_{M_{\text{sa}}}$.

COV2 If a covering $\{U_i\}_{i \in I}$ of U is a refinement of a family $\{V_j\}_{j \in J}$ in $\text{Op}_{M_{\text{sa}}}$ with $V_j \subset U$ for all $j \in J$, then $\{V_j\}_{j \in J}$ is a covering of U .

COV3 If $V \subset U$ are in $\text{Op}_{M_{\text{sa}}}$ and $\{U_i\}_{i \in I}$ is a covering of U , then $\{V \cap U_i\}_{i \in I}$ is a covering of V .

COV4 If $\{U_i\}_{i \in I}$ is a covering of U and $\{V_j\}_{j \in J}$ is a small family in $\text{Op}_{M_{\text{sa}}}$ with $V_j \subset U$ such that $\{U_i \cap V_j\}_{j \in J}$ is a covering of U_i for all $i \in I$, then $\{V_j\}_{j \in J}$ is a covering of U .

Proof. We shall use the obvious fact stating that for two subsets $A \subset B$ in M , we have $d(x, M \setminus A) \leq d(x, M \setminus B)$.

COV1 is trivial.

COV2 Let $I_0 \subset I$ be as in (1.3). Let $\sigma: I \rightarrow J$ be such that $U_i \subset V_{\sigma(i)}$, for all $i \in I$. Then, for all $x \in U_i$ we have $d(x, M \setminus U_i) \leq d(x, M \setminus V_{\sigma(i)})$. It follows that $\sigma(I_0)$ satisfies (1.3) with respect to $\{V_j\}_{j \in J}$.

COV3 Let $I_0 \subset I$ be as in (1.3) and let C be the constant in (1.2). Let x be a given point in $V \cap U$. We have $d(x, M \setminus (V \cap U)) \leq d(x, M \setminus U)$. We distinguish two cases.

(a) We assume that $d(x, M \setminus (V \cap U_i)) = d(x, M \setminus U_i)$, for all $i \in I_0$. Then we clearly have $d(x, M \setminus (V \cap U)) \leq C \max_{i \in I_0} d(x, M \setminus (V \cap U_i))$ and I_0 satisfies (1.3) with respect to $\{V \cap U_i\}_{i \in I}$.

(b) We assume $d(x, M \setminus (V \cap U_{i_0})) < d(x, M \setminus U_{i_0})$ for some $i_0 \in I_0$. Then there exists $y \in M \setminus (V \cap U_{i_0})$ such that $d(x, y) = d(x, M \setminus (V \cap U_{i_0}))$. We have $d(x, y) < d(x, M \setminus U_{i_0})$. We deduce that $y \in U_{i_0}$ and then that $y \in M \setminus V$. Hence $y \in M \setminus (V \cap U)$ and $d(x, M \setminus (V \cap U)) \leq d(x, y)$. Then

$$\begin{aligned} d(x, M \setminus (V \cap U)) &\leq d(x, M \setminus (V \cap U_{i_0})) \\ &\leq \max_{i \in I_0} d(x, M \setminus (V \cap U_i)). \end{aligned}$$

We obtain (1.2) for the family $\{V \cap U_i\}_{i \in I_0}$ with $C = 1$.

COV4 Let $I_0 \subset I$ be as in (1.3) and let C be the constant in (1.2). For each $i \in I_0$ let $J_i \subset J$ satisfy (1.3) with respect to U_i for the family $\{U_i \cap V_j\}_{j \in J}$

and let C_i be the corresponding constant. We set $J_0 = \bigcup_{i \in I_0} J_i$ and $B = \max\{C \cdot C_i; i \in I_0\}$. Then we have

$$\begin{aligned}
d(x, M \setminus U) &\leq C \max_{i \in I_0} d(x, M \setminus U_i) \\
&\leq C \max_{i \in I_0} (C_i \max_{j \in J_i} d(x, M \setminus (U_i \cap V_j))) \\
&\leq B \max_{i \in I_0} \max_{j \in J_i} d(x, M \setminus V_j) \\
&\leq B \max_{j \in J_0} d(x, M \setminus V_j),
\end{aligned}$$

which proves that J_0 satisfies (1.3) with respect to $\{V_j\}_{j \in J}$. Q.E.D.

As a particular case of COV4, we get that if $\{U_i\}_{i \in I}$ is a linear covering of $U \in \text{Op}_{M_{\text{sa}}}$ and $I = \bigsqcup_{\alpha \in A} I_\alpha$ is a partition of I , then setting $U_\alpha := \bigcup_{i \in I_\alpha} U_i$, $\{U_\alpha\}_{\alpha \in A}$ is a linear covering of U .

We shall also use the following:

Definition 1.5. Let $U \in \text{Op}_{M_{\text{sa}}}$. A regular covering of U is a sequence $\{U_i\}_{i \in [1, N]}$ with $1 \leq N \in \mathbb{N}$ such that $U = \bigcup_{i \in [1, N]} U_i$ and, for all $1 \leq k \leq N$, $\{U_i\}_{i \in [1, k]}$ is a linear covering of $\bigcup_{1 \leq i \leq k} U_i$.

Lemma 1.6. *We assume that the distance d is a subanalytic function on $M \times M$. Let $V \subset U$ be an inclusion in $\text{Op}_{M_{\text{sa}}}$ and let $2 > \varepsilon > 0$. We set*

$$(1.4) \quad V^{\varepsilon, U} = \{x \in M; d(x, V) < \varepsilon d(x, M \setminus U)\}$$

and $V' = U \setminus \overline{V}$. Then $U \cap \overline{V} \subset V^{\varepsilon, U} \subset U$ and $\{V', V^{\varepsilon, U}\}$ is a linear covering of U . In particular, for any $V'' \in \text{Op}_{M_{\text{sa}}}$ such that $V'' \cup V = U$, the pair $\{V'', V^{\varepsilon, U}\}$ is a linear covering of U .

Proof. (i) Since d is subanalytic, $V^{\varepsilon, U}$ is a subanalytic open subset.

(ii) The inclusions $U \cap \overline{V} \subset V^{\varepsilon, U} \subset U$ follow easily from the definition.

(iii) Let us prove that, for any $x \in M$,

$$(1.5) \quad d(x, M \setminus U) \leq \max\{2\varepsilon^{-1}d(x, M \setminus V'), 2(1 + \varepsilon^{-1})d(x, M \setminus V^{\varepsilon, U})\}.$$

(iii)-(a) We assume that $x \in U$ and that $d(x, M \setminus U) > 2\varepsilon^{-1}d(x, M \setminus V')$ (otherwise (1.5) is clear). We remark that $M \setminus V' = (M \setminus U) \cup \overline{V}$. Hence $d(x, M \setminus V') = d(x, (M \setminus U) \cup V)$. Hence there exists $y \in (M \setminus U) \cup V$ such that

$d(x, y) \leq (\varepsilon/2)d(x, M \setminus U)$. Since $\varepsilon < 2$ we have $y \in U$. Since $y \in (M \setminus U) \cup V$ we even have $y \in V$. Hence $d(x, V) \leq d(x, y) \leq (\varepsilon/2)d(x, M \setminus U)$.

(iii)-(b) Let $z \in M \setminus V^{\varepsilon, U}$. By the definition of $V^{\varepsilon, U}$ we have

$$\varepsilon d(z, M \setminus U) \leq d(z, V) \leq d(z, x) + d(x, V) \leq d(z, x) + (\varepsilon/2)d(x, M \setminus U).$$

On the other hand we have $d(z, M \setminus U) \geq d(x, M \setminus U) - d(z, x)$ and we deduce $d(x, M \setminus U) - d(z, x) \leq \varepsilon^{-1}d(z, x) + (1/2)d(x, M \setminus U)$. This gives $d(x, M \setminus U) \leq 2(1 + \varepsilon^{-1})d(z, x)$. Since this holds for all $z \in M \setminus V^{\varepsilon, U}$ we obtain (1.5) and the lemma follows. Q.E.D.

Lemma 1.7. *We assume that the distance d is a subanalytic function on $M \times M$. Let $\{U_i\}_{i=1}^N$ be a 1-regularly situated family in $\text{Op}_{M_{\text{sa}}}$ and let $C \geq 1$ be a constant satisfying (1.2). We choose $D > C$ and $1 > \varepsilon > 0$ such that $\varepsilon D < 1 - \varepsilon$. We define $U_i^0, V_i, U'_i \in \text{Op}_{M_{\text{sa}}}$ inductively on i by $U_1^0 = V_1 = U'_1 = U_1$ and*

$$\begin{aligned} U_i^0 &= \{x \in U_i; d(x, M \setminus (U_i \cup V_{i-1})) < D d(x, M \setminus U_i)\}, \\ V_i &= V_{i-1} \cup U_i^0, \\ U'_i &= (U_i^0)^{\varepsilon, V_i} \quad (\text{using the notation (1.4)}). \end{aligned}$$

Then $V_N = \bigcup_{i=1}^N U_i$ and, for all $k = 1, \dots, N$, we have $U'_k \subset U_k$, $V_k = \bigcup_{i=1}^k U'_i$ and $\{U'_i\}_{i=1}^k$ is a 1-regularly situated family in $\text{Op}_{M_{\text{sa}}}$.

Proof. (i) Let us prove that $U'_k \subset U_k$. Let $x \in U'_k$ and let us show that $x \in U_k$. By the definition (1.4) we have $x \in V_k$ and there exists $y \in U_k^0$ such that $d(x, y) < \varepsilon d(x, M \setminus V_k)$. We deduce $d(x, y) < \varepsilon(d(x, y) + d(y, M \setminus V_k))$ and then

$$(1.6) \quad d(x, y) < (\varepsilon/(1 - \varepsilon)) d(y, M \setminus V_k).$$

On the other hand we have $U_k^0 \subset U_k$, hence $V_k \subset U_k \cup V_{k-1}$. Since $y \in U_k^0$ we deduce

$$(1.7) \quad d(y, M \setminus V_k) \leq d(y, M \setminus (U_k \cup V_{k-1})) < D d(y, M \setminus U_k).$$

The inequalities (1.6), (1.7) and the hypothesis on D and ε give $d(x, y) < d(y, M \setminus U_k)$. Hence $x \in U_k$.

(ii) We have $V_i = V_{i-1} \cup U_i^0$. Hence Lemma 1.6 implies that $\{V_{i-1}, U_i^0\}$ is a covering of V_i in M_{sa} . Let us argue by induction. We immediately obtain that

$V_k = \bigcup_{i=1}^k U'_i$. Moreover, $\{V_{k-1}, U'_k\}$ being a covering of V_k , we get by using COV4 that, for all $k = 1, \dots, N$, $\{U'_i\}_{i=1}^k$ is a 1-regularly situated family in $\text{Op}_{M_{\text{sa}}}$.

(iii) Let us prove that $V_N = \bigcup_{i=1}^N U_i$. It is clear that $V_k \subset \bigcup_{i=1}^N U_i$, for all $k = 1, \dots, N$. Let $x \in \bigcup_{i=1}^N U_i$. Since $\{U_i\}_{i=1}^N$ is 1-regularly situated, there exists i_0 such that $d(x, M \setminus \bigcup_{i=1}^N U_i) \leq C d(x, M \setminus U_{i_0})$. In particular $x \in U_{i_0}$ and moreover $d(x, M \setminus (U_{i_0} \cup V_{i_0-1})) \leq C d(x, M \setminus U_{i_0}) < D d(x, M \setminus U_{i_0})$. Therefore $x \in U_{i_0}^0$. By definition $U_{i_0}^0 \subset V_{i_0} \subset V_N$. Hence $x \in V_N$ and we obtain $V_N = \bigcup_{i=1}^N U_i$. Q.E.D.

In particular, we have proved:

Proposition 1.8. *Let $U \in \text{Op}_{M_{\text{sa}}}$. Then for any linear covering $\{U_i\}_{i \in I}$ of U there exists a refinement which is a regular covering of U .*

Definition 1.9. (a) The linear subanalytic site M_{sal} is the presite M_{sa} endowed with the Grothendieck topology for which the coverings are the linear coverings given by Definition 1.3.

(b) We denote by $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$ the natural morphism of sites.

Remark 1.10. Let $f: M \rightarrow N$ be a bi-Lipschitz subanalytic homeomorphism between two real analytic manifolds. Then $f^{-1}: \text{Op}_{M_{\text{sa}}} \rightarrow \text{Op}_{N_{\text{sa}}}$ induces an isomorphism of sites $N_{\text{sal}} \xrightarrow{\sim} M_{\text{sal}}$.

We have natural functors

$$(1.8) \quad \text{Mod}(\mathbf{k}_{M_{\text{sa}}}) \xrightleftharpoons[\rho_{\text{sal}}^{-1}]{\rho_{\text{sal}*}} \text{Mod}(\mathbf{k}_{M_{\text{sal}}}), \quad \text{D}^b(\mathbf{k}_{M_{\text{sa}}}) \xrightleftharpoons[\rho_{\text{sal}}^{-1}]{\text{R}\rho_{\text{sal}*}} \text{D}^b(\mathbf{k}_{M_{\text{sal}}}).$$

Lemma 1.11. *The functor $\rho_{\text{sal}*}$ in (1.8) is fully faithful and $\rho_{\text{sal}}^{-1}\rho_{\text{sal}*} \simeq \text{id}$. Moreover, $\rho_{\text{sal}}^{-1}\text{R}\rho_{\text{sal}*} \simeq \text{id}$.*

Proof. (i) By its definition, $\rho_{\text{sal}}^{-1}\rho_{\text{sal}*}F$ is the sheaf associated with the presheaf $U \mapsto (\rho_{\text{sal}*}F)(U) \simeq F(U)$ and this presheaf is already a sheaf.

(ii) Since ρ_{sal}^{-1} is exact, $\rho_{\text{sal}}^{-1}\text{R}\rho_{\text{sal}*}$ is the derived functor of $\rho_{\text{sal}}^{-1}\rho_{\text{sal}*}$. Q.E.D.

Sheaves on M_{sa} and M_{sal}

Proposition 1.12. *Let $U \in \text{Op}_{M_{\text{sa}}}$. Then $\rho_{\text{sal}*} \mathbf{k}_{UM_{\text{sa}}} \simeq \mathbf{k}_{UM_{\text{sal}}}$ and $\rho_{\text{sal}}^{-1} \mathbf{k}_{UM_{\text{sal}}} \simeq \mathbf{k}_{UM_{\text{sa}}}$.*

Proof. The proof of [KS01, Prop. 6.3.1] gives the first isomorphism without any changes other than notational. The second isomorphism follows by Lemma 1.11. Q.E.D.

Remark 1.13. Denote by $M_{\text{sa}0}$ the site for which the open sets are those of M_{sa} but a family $\{U_i\}_{i \in I}$ of open subsets of U is a covering of U if and only if there exists i with $U_i = U$. Then the sheaves on $M_{\text{sa}0}$ are nothing but the presheaves on M_{sa} and one may ask why to consider M_{sal} and not $M_{\text{sa}0}$ which is easier to manipulate. One reason is that Proposition 1.12 is no more true with this new site, and, as a by-product, Theorem 3.11 below would no more be true with $M_{\text{sa}0}$ instead of M_{sal} .

Proposition 1.14. *Let $U \in \text{Op}_{M_{\text{sa}}}$ and let $F \in \text{Mod}(\mathbf{k}_{M_{\text{sa}}})$. Then*

$$\text{R}\Gamma(U; \text{R}\rho_{\text{sal}*} F) \simeq \text{R}\Gamma(U; F).$$

Proof. This follows from $\text{R}\Gamma(U; G) \simeq \text{RHom}(\mathbf{k}_U, G)$ for $G \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$ ($\mathcal{T} = M_{\text{sa}}$ or $\mathcal{T} = M_{\text{sal}}$) and by adjunction since $\rho_{\text{sal}}^{-1} \mathbf{k}_{UM_{\text{sal}}} \simeq \mathbf{k}_{UM_{\text{sa}}}$. Q.E.D.

In the sequel we shall simply denote by \mathbf{k}_U the sheaf $\mathbf{k}_{U\mathcal{T}}$ for $\mathcal{T} = M_{\text{sa}}$ or $\mathcal{T} = M_{\text{sal}}$.

Proposition 1.15. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} . Then a presheaf F is a sheaf if and only if it satisfies:*

- (i) $F(\emptyset) = 0$,
- (ii) *for any $U_1, U_2 \in \text{Op}_{M_{\text{sa}}}$ such that $\{U_1, U_2\}$ is a covering of $U_1 \cup U_2$, the sequence $0 \rightarrow F(U_1 \cup U_2) \rightarrow F(U_1) \oplus F(U_2) \rightarrow F(U_1 \cap U_2)$ is exact.*

Of course, if $\mathcal{T} = M_{\text{sa}}$, $\{U_1, U_2\}$ is always a covering of $U_1 \cup U_2$.

Proof. In the case of the site M_{sa} this is Proposition 6.4.1 of [KS01]. Let F be a presheaf on M_{sal} such that (i) and (ii) are satisfied and let us prove that F is a sheaf. Let $U \in \text{Op}_{M_{\text{sa}}}$ and let $\{U_i\}_{i \in I}$ be a linear covering of U . By Proposition 1.8 we can find a finite refinement $\{V_j\}_{j \in J}$ of $\{U_i\}_{i \in I}$ which is

a regular covering of U . We choose $\sigma: J \rightarrow I$ such that $V_j \subset U_{\sigma(j)}$, for all $j \in J$, and we consider the commutative diagram

$$(1.9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F(U) & \xrightarrow{u} & \bigoplus_{i \in I} F(U_i) & \xrightarrow{v} & \bigoplus_{i,j \in I} F(U_{ij}) \\ & & \parallel & & \downarrow a & & \downarrow b \\ 0 & \longrightarrow & F(U) & \longrightarrow & \bigoplus_{k \in J} F(V_k) & \longrightarrow & \bigoplus_{k,l \in J} F(V_{kl}), \end{array}$$

where a and b are defined as follows. For $s = \{s_i\}_{i \in I} \in \bigoplus_{i \in I} F(U_i)$, we set $a(s) = \{t_k\}_{k \in J} \in \bigoplus_{k \in J} F(V_k)$ where $t_k = s_{\sigma(k)}|_{V_k}$. In the same way we set $b(\{s_{ij}\}_{i,j \in I}) = \{s_{\sigma(k)\sigma(l)}|_{V_{kl}}\}_{k,l \in J}$. The proof of [KS01, Prop. 6.4.1] applies to a regular covering in M_{sal} and we deduce that the bottom row of the diagram (1.9) is exact. It follows immediately that $\text{Ker } u = 0$. This proves that F is a separated presheaf.

It remains to prove that $\text{Ker } v = \text{Im } u$. Let $s = \{s_i\}_{i \in I} \in \bigoplus_{i \in I} F(U_i)$ be such that $v(s) = 0$. By the exactness of the bottom row we can find $t \in F(U)$ such that $a(u(t) - s) = 0$. Let us check that $t|_{U_i} = s_i$ for any given $i \in I$. The family $\{U_i \cap V_k\}_{k \in J}$ is a covering of U_i in M_{sal} . Since F is separated it is enough to see that $t|_{U_i \cap V_k} = s_i|_{U_i \cap V_k}$ for all $k \in J$. Setting $W = U_i \cap V_k$, we have

$$t|_W = s_{\sigma(k)}|_W = (s_{\sigma(k)}|_{U_i \cap U_{\sigma(k)}})|_W = (s_i|_{U_i \cap U_{\sigma(k)}})|_W = s_i|_W,$$

where the first equality follows from $a(u(t) - s) = 0$ and the third one from $v(s) = 0$. Q.E.D.

Lemma 1.16. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} . Let $U \in \text{Op}_{M_{\text{sa}}}$ and let $\{F_i\}_{i \in I}$ be an inductive system in $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ indexed by a small filtrant category I . Then*

$$(1.10) \quad \varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i).$$

This kind of results is well-known from the specialists (see *e.g.* [KS01, EP10]) but for the reader's convenience, we give a proof.

Proof. For a covering $\mathcal{S} = \{U_j\}_j$ of U set

$$\Gamma(\mathcal{S}; F) := \text{Ker}\left(\prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \cap U_j)\right).$$

Denote by “ \varinjlim ” the inductive limit in the category of presheaves and recall that $\varinjlim_i F_i$ is the sheaf associated with “ \varinjlim_i ” F_i . The presheaf “ \varinjlim_i ” F_i is separated. Denote by $\text{Cov}(U)$ the family of coverings of U in \mathcal{T} ordered as follows. For \mathcal{S}_1 and \mathcal{S}_2 in $\text{Cov}(U)$, $\mathcal{S}_1 \preceq \mathcal{S}_2$ if \mathcal{S}_1 is a refinement of \mathcal{S}_2 . Then $\text{Cov}(U)^{\text{op}}$ is filtrant and

$$\begin{aligned} \Gamma(U; \varinjlim_i F_i) &\simeq \varinjlim_{\mathcal{S} \in \text{Cov}(U)} \Gamma(\mathcal{S}; \varinjlim_i F_i) \\ &\simeq \varinjlim_{\mathcal{S}} \varinjlim_i \Gamma(\mathcal{S}; F_i) \\ &\simeq \varinjlim_i \varinjlim_{\mathcal{S}} \Gamma(\mathcal{S}; F_i) \simeq \varinjlim_i \Gamma(U; F_i). \end{aligned}$$

Here, the second isomorphism follows from the fact that we may assume that the covering \mathcal{S} is finite. Q.E.D.

Example 1.17. Let $M = \mathbb{R}^2$ endowed with coordinates $x = (x_1, x_2)$. For $\varepsilon, A > 0$ we define the subanalytic open subset

$$(1.11) \quad U_{A,\varepsilon} = \{x; 0 < x_1 < \varepsilon, -Ax_1^2 < x_2 < Ax_1^2\}.$$

We define a presheaf F on M_{sal} by setting, for any $V \in \text{Op}_{M_{\text{sa}}}$,

$$F(V) = \begin{cases} \mathbf{k} & \text{if for any } A > 0, \text{ there exists } \varepsilon > 0 \text{ such that } U_{A,\varepsilon} \subset V, \\ 0 & \text{otherwise.} \end{cases}$$

The restriction map $F(V) \rightarrow F(V')$, for $V' \subset V$, is $\text{id}_{\mathbf{k}}$ if $F(V') = \mathbf{k}$. We prove that F is sheaf in (iii) below after the preliminary remarks (i) and (ii).

(i) For a given $A > 0$ we have $d((\varepsilon/2, 0), M \setminus U_{A,\varepsilon}) \geq (A/16)\varepsilon^2$, for any $\varepsilon > 0$ small enough. In particular, if $F(V) = \mathbf{k}$, then

$$(1.12) \quad d((\varepsilon, 0), M \setminus V)/\varepsilon^2 \rightarrow +\infty \quad \text{when } \varepsilon \rightarrow 0.$$

(ii) Let us assume that there exist $A > 0$ and a sequence $\{\varepsilon_n\}$, $n \in \mathbb{N}$, such that $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$ and V contains the closed balls $\overline{B((\varepsilon_n, 0), A\varepsilon_n^2)}$ for all $n \in \mathbb{N}$. Then there exists $\varepsilon > 0$ such that V contains $\overline{U_{A,\varepsilon}} \setminus \{0\}$.

Before we prove this claim we translate the conclusion in terms of sheaf theory (in the usual site \mathbb{R}^2). Let $p: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection $(x_1, x_2) \mapsto x_1$.

Then, for $x_1 > 0$, the set $p^{-1}(x_1) \cap V \cap \overline{U_{A,\varepsilon}}$ is a finite disjoint union of intervals, say I_1, \dots, I_N . If $p^{-1}(x_1) \cap V$ contains $p^{-1}(x_1) \cap \overline{U_{A,\varepsilon}}$, then $N = 1$, I_1 is closed and $R\Gamma(\mathbb{R}; \mathbf{k}_{I_1}) = \mathbf{k}$. In the other case none of these I_1, \dots, I_N is closed and $H^0(\mathbb{R}; \mathbf{k}_{I_j}) = 0$, for all $j = 1, \dots, N$. By the base change formula we deduce that V contains $\overline{U_{A,\varepsilon}} \setminus \{0\}$ if and only if $Rp_*(\mathbf{k}_{V \cap \overline{U_{A,\varepsilon}}})|_{]0,\varepsilon]} \simeq \mathbf{k}_{]0,\varepsilon]}$.

We remark that, for $\varepsilon < 1$, we have $Rp_*(\mathbf{k}_{V \cap \overline{U_{A,\varepsilon}}})|_{]0,\varepsilon]} \simeq Rp_*(\mathbf{k}_{V \cap \overline{U_{A,1}}})|_{]0,\varepsilon]}$. The sheaf $Rp_*(\mathbf{k}_{V \cap \overline{U_{A,1}}})$ is constructible. Hence it is constant on $]0, \varepsilon]$ for $\varepsilon > 0$ small enough. Since $(Rp_*(\mathbf{k}_{V \cap \overline{U_{A,1}}}))_{\varepsilon_n} \simeq \mathbf{k}$ by hypothesis, the conclusion follows.

(iii) Now we check that F is a sheaf on M_{sal} with the criterion of Proposition 1.15. Let $U, U_1, U_2 \in \text{Op}_{M_{\text{sa}}}$ such that $\{U_1, U_2\}$ is a covering of U . We let $C > 0$ be a constant satisfying (1.2).

(iii-a) Let us prove that $F(U) \rightarrow F(U_1) \oplus F(U_2)$ is injective. So we assume that $F(U) = \mathbf{k}$ (otherwise this is obvious) and we prove that $F(U_1) = \mathbf{k}$ or $F(U_2) = \mathbf{k}$. Let $A > 0$. By (1.12) and (1.2) there exists $\varepsilon_0 > 0$ such that

$$\max\{d((\varepsilon, 0), M \setminus U_1), d((\varepsilon, 0), M \setminus U_2)\} \geq (A/C)\varepsilon^2, \quad \text{for all } \varepsilon \in]0, \varepsilon_0[.$$

Hence, for any integer $n \geq 1$, the ball $B((1/n, 0), A/(Cn^2))$ is included in U_1 or U_2 . One of U_1 or U_2 must contain infinitely many such balls. By (ii) we deduce that it contains $U_{A/C, \varepsilon_A}$, for some $\varepsilon_A > 0$. When A runs over \mathbb{N} we deduce that one of U_1 or U_2 contains infinitely many sets of the type $U_{A/C, \varepsilon_A}$, $A \in \mathbb{N}$. Hence $F(U_1) = \mathbf{k}$ or $F(U_2) = \mathbf{k}$.

(iii-b) Now we prove that the kernel of $F(U_1) \oplus F(U_2) \rightarrow F(U_{12})$ is $F(U)$. We see easily that the only case where this kernel could be bigger than $F(U)$ is $F(U_1) = F(U_2) = \mathbf{k}$ and $F(U_{12}) = 0$. In this case, for any $A > 0$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $U_{A, \varepsilon_1} \subset U_1$ and $U_{A, \varepsilon_2} \subset U_2$. This gives $U_{A, \min\{\varepsilon_1, \varepsilon_2\}} \subset U_{12}$ which contradicts $F(U_{12}) = 0$.

(iv) By the definition of F we have a natural morphism $u: F \rightarrow \rho_{\text{sal}*} \mathbf{k}_{\{0\}}$ which is surjective. We can see that $\rho_{\text{sal}}^{-1}(u)$ is an isomorphism. We define $N \in \text{Mod}(\mathbf{k}_{M_{\text{sal}}})$ by the exact sequence

$$(1.13) \quad 0 \rightarrow N \rightarrow F \rightarrow \rho_{\text{sal}*} \mathbf{k}_{\{0\}} \rightarrow 0.$$

Then $\rho_{\text{sal}}^{-1}N \simeq 0$ but $N \neq 0$. More precisely, for $V \in \text{Op}_{M_{\text{sa}}}$, we have $N(V) = 0$ if $0 \in V$ and $N(V) \xrightarrow{\sim} F(V)$ if $0 \notin V$.

2 Γ -acyclic sheaves

Cech complexes

In this subsection, \mathcal{T} denotes either the site M_{sa} or the site M_{sal} .

For a finite set I and a family of open subset $\{U_i\}_{i \in I}$ we set for $\emptyset \neq J \subset I$,

$$U_J := \bigcap_{j \in J} U_j.$$

Lemma 2.1. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} . Let $\{U_1, U_2\}$ be a covering of $U_1 \cup U_2$. Then the sequence*

$$(2.1) \quad 0 \rightarrow \mathbf{k}_{U_{12}} \rightarrow \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \rightarrow \mathbf{k}_{U_1 \cup U_2} \rightarrow 0$$

is exact.

Proof. The result is well-known for the site M_{sa} and the functor $\rho_{\text{sal}*}$ being left exact, it remains to show that $\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \rightarrow \mathbf{k}_{U_1 \cup U_2}$ is an epimorphism. This follows from the fact that for any $F \in \text{Mod}(\mathbf{k}_{M_{\text{sal}}})$, the map $\text{Hom}_{\mathbf{k}_{M_{\text{sal}}}}(\mathbf{k}_{U_1 \cup U_2}, F) \rightarrow \text{Hom}_{\mathbf{k}_{M_{\text{sal}}}}(\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2}, F)$ is a monomorphism. Q.E.D.

Consider now a finite family $\{U_i\}_{i \in I}$ of objects of $\text{Op}_{M_{\text{sa}}}$ and let $N := |I|$. Then we have the Cech complex in $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ in which the term corresponding to $|J| = 1$ is in degree 0.

$$(2.2) \quad \mathbf{k}_{\mathcal{U}}^\bullet := 0 \rightarrow \bigoplus_{\emptyset \neq J \subset I, |J|=N} \mathbf{k}_{U_J} \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{\emptyset \neq J \subset I, |J|=1} \mathbf{k}_{U_J} \otimes e_J \rightarrow 0.$$

Recall that $\{e_J\}_{|J|=k}$ is a basis of $\bigwedge^k \mathbb{Z}^N$ and the differential is defined as usual by sending $\mathbf{k}_{U_J} \otimes e_J$ to $\bigoplus_{i \in I} \mathbf{k}_{U_{J \setminus i}} \otimes e_i \rfloor e_J$ using the natural morphism $\mathbf{k}_{U_J} \rightarrow \mathbf{k}_{U_{J \setminus i}}$.

Proposition 2.2. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} . Let $U \in \text{Op}_{M_{\text{sa}}}$ and let $\mathcal{U} := \{U_i\}_{i \in I}$ be a finite covering of U in \mathcal{T} (a regular covering in case $\mathcal{T} = M_{\text{sal}}$). Then the natural morphism $\mathbf{k}_{\mathcal{U}}^\bullet \rightarrow \mathbf{k}_U$ is a quasi-isomorphism.*

Proof. Let $N = |I|$. We may assume $I = [1, N]$. For $N = 2$ this is nothing but Lemma 2.1. We argue by induction and assume the result is proved

for $N - 1$. Denote by \mathcal{U}' the covering of $U' := \bigcup_{1 \leq i \leq N-1} U_i$ by the family $\{U_i\}_{i \in [1, \dots, N-1]}$. Consider the subcomplex F_1 of $\mathbf{k}_{\mathcal{U}}^\bullet$ given by

$$(2.3) \quad F_1 := 0 \rightarrow \bigoplus_{N \in J \subset I, |J|=N} \mathbf{k}_{U_J} \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{N \in J \subset I, |J|=1} \mathbf{k}_{U_J} \otimes e_J \rightarrow 0.$$

Note that F_1 is isomorphic to the complex $\mathbf{k}_{\mathcal{U}' \cap U_N}^\bullet \rightarrow \mathbf{k}_{U_N}$ where \mathbf{k}_{U_N} is in degree 0 and we shall represent F_1 by this last complex. By [KS06, Th. 12.4.3], there are natural morphisms of complexes

$$(2.4) \quad \mathbf{k}_{\mathcal{U}'}^\bullet[-1] \rightarrow \mathbf{k}_{\mathcal{U}}^\bullet[-1] \xrightarrow{d} \mathbf{k}_{\mathcal{U}}^\bullet \rightarrow (\mathbf{k}_{\mathcal{U}' \cap U_N}^\bullet \rightarrow \mathbf{k}_{U_N})$$

and $\mathbf{k}_{\mathcal{U}}^\bullet$ is isomorphic to the mapping cone of the morphism

$$(2.5) \quad \mathbf{k}_{\mathcal{U}'}^\bullet[-1] \xrightarrow{u} (\mathbf{k}_{\mathcal{U}' \cap U_N}^\bullet \rightarrow \mathbf{k}_{U_N}).$$

Hence, writing the long exact sequence associated with the mapping cone of u , we are reduced, by the induction hypothesis, to prove that the morphism

$$\mathbf{k}_{U' \cap U_N} \rightarrow \mathbf{k}_{U'} \oplus \mathbf{k}_{U_N}$$

is a monomorphism and its cokernel is isomorphic to \mathbf{k}_U . Since $\{U', U_N\}$ is a covering of U , this follows from Lemma 2.1. Q.E.D.

Acyclic sheaves

In this subsection, \mathcal{T} denotes either the site M_{sa} or the site M_{sal} . In the literature, one often encounters sheaves which are $\Gamma(U; \bullet)$ -acyclic for a given $U \in \mathcal{T}$ but the next definition does not seem to be frequently used.

Definition 2.3. Let $F \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$. We say that F is Γ -acyclic if $H^k(U; F) \simeq 0$ for all $k > 0$ and all $U \in \mathcal{T}$.

We shall give criteria in order that a sheaf F on the site \mathcal{T} be Γ -acyclic.

Let $U \in \text{Op}_{M_{\text{sa}}}$ and let $\mathcal{U} := \{U_i\}_{i \in I}$ be a finite covering of U in \mathcal{T} (a regular covering in case $\mathcal{T} = M_{\text{sal}}$). We denote by $C^\bullet(\mathcal{U}; F)$ the associated Čech complex:

$$(2.6) \quad C^\bullet(\mathcal{U}; F) := \text{Hom}_{\mathbf{k}_{M_{\text{sal}}}}(\mathbf{k}_{\mathcal{U}}^\bullet, F).$$

One can write more explicitly this complex as the complex:

$$(2.7) \quad 0 \rightarrow \bigoplus_{\emptyset \neq J \subset I, |J|=1} F(U_J) \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{\emptyset \neq J \subset I, |J|=N} F(U_J) \otimes e_J \rightarrow 0$$

where the differential d is obtained by sending $F(U_J) \otimes e_J$ to $\bigoplus_{i \in I} F(U_J \cap U_i) \otimes e_i \wedge e_J$.

Proposition 2.4. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} and let $F \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$. The conditions below are equivalent.*

- (i) *For any $\{U_1, U_2\}$ which is a covering of $U_1 \cup U_2$, the sequence $0 \rightarrow F(U_1 \cup U_2) \rightarrow F(U_1) \oplus F(U_2) \rightarrow F(U_1 \cap U_2) \rightarrow 0$ is exact.*
- (ii) *For any finite covering \mathcal{U} of U (regular covering in case $\mathcal{T} = M_{\text{sal}}$), the morphism $F(U) \rightarrow C^\bullet(\mathcal{U}; F)$ is a quasi-isomorphism.*
- (iii) *The sheaf F is Γ -acyclic.*
- (iv) *For any exact sequence in $\text{Mod}(\mathbf{k}_{\mathcal{T}})$*

$$(2.8) \quad G^\bullet := 0 \rightarrow \bigoplus_{i_0 \in A_0} \mathbf{k}_{U_{i_0}} \rightarrow \cdots \rightarrow \bigoplus_{i_N \in A_N} \mathbf{k}_{U_{i_N}} \rightarrow 0$$

the sequence $\text{Hom}_{\mathbf{k}_{\mathcal{T}}}(G^\bullet, F)$ is exact.

Proof. (i) \Rightarrow (ii) is proved by induction on the cardinal of I as in the proof of Proposition 2.2. Details are left to the reader.

(ii) \Rightarrow (iii) (a) Let $U \in \text{Op}_{M_{\text{sa}}}$. Let us first show that for any exact sequence of sheaves $0 \rightarrow F \xrightarrow{\varphi} F' \xrightarrow{\psi} F'' \rightarrow 0$ and any $U \in \text{Op}_{M_{\text{sa}}}$, the sequence $0 \rightarrow F(U) \rightarrow F'(U) \rightarrow F''(U) \rightarrow 0$ is exact. Let $s'' \in F''(U)$. By the exactness of the sequence of sheaves, there exists a finite covering $U = \bigcup_{i=1}^N U_i$ and $s'_i \in F'(U_i)$ such that $\psi(s'_i) = s''|_{U_i}$. In case $\mathcal{T} = M_{\text{sal}}$, we may assume that the covering is regular by Proposition 1.8. For $k = 1, \dots, N$, we set $V_k = \bigcup_{i=1}^k U_i$. Let us prove by induction on k that there exists $t'_k \in F'(V_k)$ such that $\psi(t'_k) = s''|_{V_k}$. Starting with $t'_1 = s'_1$ we assume that we have found t'_k . Since our covering is regular, $\{V_k, U_{k+1}\}$ is a covering of V_{k+1} . We set for short $W = V_k \cap U_{k+1}$. We have $\psi(t'_k|_W) = \psi(s'_{k+1}|_W)$. Hence there exists $s \in F(W)$ such that $\varphi(s) = t'_k|_W - s'_{k+1}|_W$. By hypothesis (ii) there exists $s_V \in F(V_k)$ and $s_U \in F(U_{k+1})$ such that $s = s_V|_W - s_U|_W$.

Setting $t'_V = t'_k - \psi(s_V)$ and $s'_U = s'_{k+1} - \psi(s_U)$ we obtain $t'_U|_W = s'_V|_W$ and we can glue $t'_U|_W$ and $s'_V|_W$ into $t'_{k+1} \in F(V_{k+1})$. We check easily that $\psi(t'_{k+1}) = s''|_{V_{k+1}}$ and the induction proceeds.

(ii) \Rightarrow (iii) (b) Denote by \mathcal{J} the full additive subcategory of $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ consisting of sheaves satisfying the condition (i). We shall show that the category \mathcal{J} is $\Gamma(U; \bullet)$ -injective for all $U \in \text{Op}_{M_{\text{sa}}}$. Let $F^\bullet := 0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be a short exact sequence of sheaves.

The category \mathcal{J} contains the injective sheaves. By the preceding result, it thus remains to show that if both F' and F belong to \mathcal{J} , then F'' belongs to \mathcal{J} .

Let U_1, U_2 as in (i) and denote by $\mathbf{k}_{\mathcal{U}}^\bullet$ the exact sequence $0 \rightarrow \mathbf{k}_{U_1 \cap U_2} \rightarrow \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \rightarrow \mathbf{k}_{U_1 \cup U_2} \rightarrow 0$. Consider the double complex $\text{Hom}_{\mathbf{k}_{\mathcal{T}}}(\mathbf{k}_{\mathcal{U}}^\bullet, F^\bullet)$. By the preceding result all rows and columns except at most one (either one row or one column depending how one writes the double complex) are exact. It follows that the double complex is exact.

(iii) \Rightarrow (iv) Consider an injective resolution I^\bullet of F , that is, a complex I^\bullet of injective sheaves such that the sequence $I^{\bullet,+} := 0 \rightarrow F \rightarrow I^\bullet$ is exact. The hypothesis implies that $\Gamma(W; I^{\bullet,+})$ remains exact for all $W \in \text{Op}_{M_{\text{sa}}}$. Then the argument goes as in the proof of (ii) \Rightarrow (iii) (b). Recall that G^\bullet denotes the complex of (2.8) and consider the double complex $\text{Hom}_{\mathbf{k}_{\mathcal{T}}}(G^\bullet, I^{\bullet,+})$. Then all its rows and columns except one (either one row or one column depending how one writes the double complex) will be exact. It follows that all rows and columns are exact.

(iv) \Rightarrow (i) is obvious by Lemma 2.1.

Q.E.D.

Corollary 2.5. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} . A small filtrant inductive limit of Γ -acyclic sheaves is Γ -acyclic.*

Proof. Since small filtrant inductive limits are exact in $\text{Mod}(\mathbf{k})$, the family of sheaves satisfying condition (i) of Proposition 2.4 is stable by such limits by Lemma 1.16.

Q.E.D.

Definition 2.6. Let \mathcal{T} be either the site M_{sa} or the site M_{sal} . One says that $F \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$ is flabby if for any U and V in $\text{Op}_{M_{\text{sa}}}$ with $V \subset U$, the natural morphism $F(U) \rightarrow F(V)$ is surjective.

Lemma 2.7. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} .*

- (i) *Injective sheaves are flabby.*
- (ii) *Flabby sheaves are Γ -acyclic.*
- (iii) *The category of flabby sheaves is stable by small filtrant inductive limits.*

Proof. (i) Let F be an injective sheaf and let U and V in $\text{Op}_{M_{\text{sa}}}$ with $V \subset U$. Recall that the sequence $0 \rightarrow \mathbf{k}_V \rightarrow \mathbf{k}_U$ is exact. Applying the functor $\text{Hom}_{\mathbf{k}_{\mathcal{T}}}(\cdot, F)$ we get the result.

(ii) If $F \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$ is flabby then it satisfies condition (i) of Proposition 2.4.

(iii) Let $\{F_i\}_{i \in I}$ be a small filtrant inductive system of flabby objects in $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ and let U and V in $\text{Op}_{M_{\text{sa}}}$ with $V \subset U$. The family of epimorphisms $F_i(U) \twoheadrightarrow F_i(V)$ gives the epimorphism $\varinjlim_i F_i(U) \twoheadrightarrow \varinjlim_i F_i(V)$. Applying Lemma 1.16 we get the epimorphism $\Gamma(U; \varinjlim_i F_i) \twoheadrightarrow \Gamma(V; \varinjlim_i F_i)$. Q.E.D.

3 The functor $\rho_{\text{sal}}^!$

Lemma 3.1. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} and let $U \in \text{Op}_{M_{\text{sa}}}$. Let I be a small filtrant category and $\alpha: I \rightarrow \text{Mod}(\mathbf{k}_{\mathcal{T}})$ a functor. Set for short $F_i = \alpha(i)$. Then for any $j \in \mathbb{Z}$*

$$(3.1) \quad H^j \text{R}\Gamma(U; \varinjlim_i F_i) \simeq \varinjlim_i H^j \text{R}\Gamma(U; F_i).$$

Proof. (i) Denote by \mathcal{I} the full additive subcategory of $\text{Mod}(\mathbf{k}_{\mathcal{T}})$ consisting of injective sheaves. It follows for example from [KS06, Cor. 9.6.6] that there exists a functor $\psi: I \rightarrow \mathcal{I}$ and a morphism of functors $\alpha \rightarrow \psi$ such that for each $i \in I$, $\alpha(i) \rightarrow \psi(i)$ is a monomorphism. Therefore one can construct a functor $\Psi: I \rightarrow \text{C}^+(\mathcal{I})$ (recall that $\text{C}^+(\mathcal{I})$ is the additive category of bounded from below complexes of \mathcal{I}) and a morphism of functor $\alpha \rightarrow \Psi$ such that for each $i \in I$, $\alpha(i) \rightarrow \Psi(i)$ is a quasi-isomorphism.

(ii) Set for short $G_i^\bullet = \Psi(i)$. Applying Lemma 1.16, we get

$$\begin{aligned} \varinjlim_i H^j \text{R}\Gamma(U; F_i) &\simeq \varinjlim_i H^j \Gamma(U; G_i^\bullet) \\ &\simeq H^j \Gamma(U; \varinjlim_i G_i^\bullet). \end{aligned}$$

The complex $\varinjlim_i G_i^\bullet$ is a complex of flabby sheaves by Lemma 2.7. Therefore $\varinjlim_i G_i^\bullet$ is a $\Gamma(U; \bullet)$ -injective resolution of $\varinjlim_i F_i$ which implies

$$H^j \Gamma(U; \varinjlim_i G_i^\bullet) \simeq H^j \mathrm{R}\Gamma(U; \varinjlim_i F_i).$$

Q.E.D.

Lemma 3.2. *Let \mathcal{C} be a Grothendieck category and let $d \in \mathbb{Z}$. Then the cohomology functor H^d and the truncation functors $\tau^{\leq d}$ and $\tau^{\geq d}$ commute with small direct sums in $\mathrm{D}(\mathcal{C})$. In other words, if $\{F_i\}_{i \in I}$ is a small family of objects of $\mathrm{D}(\mathcal{C})$, then*

$$(3.2) \quad \bigoplus_i \tau^{\leq d} F_i \xrightarrow{\sim} \tau^{\leq d} \left(\bigoplus_i F_i \right)$$

and similarly with $\tau^{\geq d}$ and H^d .

Proof. (i) The case of H^d follows from [KS06, Prop. 10.2.8, Prop. 14.1.1].
(ii) The morphism in (3.2) is well-defined and it is enough to check that it induces an isomorphism on the cohomology. This follows from (i) since for any object $Y \in \mathrm{D}(\mathcal{C})$, $H^j(\tau^{\leq d} Y)$ is either 0 or $H^j(Y)$. Q.E.D.

Lemma 3.3. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} and let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. Let $-\infty < a \leq b < \infty$, let I be a small set and let $F_i \in \mathrm{D}^{[a,b]}(\mathbf{k}_{\mathcal{T}})$. Then*

$$(3.3) \quad \bigoplus_i \mathrm{R}\Gamma(U; F_i) \xrightarrow{\sim} \mathrm{R}\Gamma(U; \bigoplus_i F_i).$$

Proof. The morphism in (3.3) is well-defined and we have to prove it is an isomorphism. If $b = a + 1$, the result follows from Lemma 3.1. The general case is deduced by induction on $b - a$ by considering the distinguished triangles

$$H^a(F_i)[-a] \rightarrow F_i \rightarrow \tau^{\geq a+1} F_i \xrightarrow{+1}$$

and using the fact that H^a and $\tau^{\geq a+1}$ commute with direct sums by Lemma 3.2. Q.E.D.

Lemma 3.4. *Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$. The functor $\Gamma(U; \bullet): \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \rightarrow \mathrm{Mod}(\mathbf{k})$ has cohomological dimension $\dim M$.*

Proof. We know that if $F \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$, then $H^j \text{R}\Gamma(U; F) \simeq 0$ for $j > \dim M$. Since any $F \in \text{Mod}(\mathbf{k}_{M_{\text{sa}}})$ is a small filtrant inductive limit of constructible sheaves, the result follows from Lemma 3.1. Q.E.D.

Proposition 3.5. *Let \mathcal{J} be the subcategory of $\text{Mod}(\mathbf{k}_{M_{\text{sa}}})$ consisting of sheaves which are Γ -acyclic. For any $F \in \text{Mod}(\mathbf{k}_{M_{\text{sa}}})$, there exists an exact sequence $0 \rightarrow F \rightarrow F^0 \rightarrow \cdots \rightarrow F^n \rightarrow 0$ where $n = \dim M$ and the F^j 's belong to \mathcal{J} .*

Proof. Consider a resolution $0 \rightarrow F \rightarrow I^0 \xrightarrow{d^0} I^1 \rightarrow \cdots$ with the I^j 's injective and define $F^j = I^j$ for $j \leq n-1$, $F^j = 0$ for $j > n$ and $F^n = \text{Ker } d^n$. It follows from Lemma 3.4 that F^n is Γ -acyclic. Q.E.D.

Proposition 3.6. *Let I be a small set and let $F_i \in \text{D}(\mathbf{k}_{M_{\text{sa}}})$ ($i \in I$). For $U \in \text{Op}_{M_{\text{sa}}}$, we have the natural isomorphism*

$$(3.4) \quad \bigoplus_{i \in I} \text{R}\Gamma(U; F_i) \xrightarrow{\sim} \text{R}\Gamma(U; \bigoplus_{i \in I} F_i) \text{ in } \text{D}(\mathbf{k}).$$

Proof. Since the morphism in (3.4) is well-defined, it is enough to check that it induces an isomorphism on the cohomology groups.

Let $n = \dim M$. For $G \in \text{D}(\mathbf{k}_{M_{\text{sa}}})$ and for $j \in \mathbb{Z}$, we have by Lemma 3.4

$$\tau^{\geq j} \text{R}\Gamma(U; G) \simeq \tau^{\geq j} \text{R}\Gamma(U; \tau^{\geq j-n-1} G).$$

The functor $\Gamma(U; \bullet)$ being left exact, for $k \geq j$ we get

$$(3.5) \quad H^k \text{R}\Gamma(U; G) \simeq H^k \text{R}\Gamma(U; \tau^{\leq k} \tau^{\geq j-n-1} G).$$

We have the sequence of isomorphisms:

$$\begin{aligned} H^k \text{R}\Gamma(U; \bigoplus_i F_i) &\simeq H^k \text{R}\Gamma(U; \tau^{\leq k} \tau^{\geq j-n-1} \bigoplus_i F_i) \\ &\simeq H^k \text{R}\Gamma(U; \bigoplus_i \tau^{\leq k} \tau^{\geq j-n-1} F_i) \\ &\simeq \bigoplus_i H^k \text{R}\Gamma(U; \tau^{\leq k} \tau^{\geq j-n-1} F_i) \\ &\simeq \bigoplus_i H^k \text{R}\Gamma(U; F_i). \end{aligned}$$

The first and last isomorphisms follow from (3.5).

The second isomorphism follows from Lemma 3.2.

The third isomorphism follows from (3.3) in Lemma 3.3. Q.E.D.

The functor $R\rho_{\text{sal}*}$

Lemma 3.7. *Let \mathcal{T} be either the site M_{sa} or the site M_{sal} and let $U \in \text{Op}_{M_{\text{sa}}}$. Let $-\infty < a \leq b < \infty$, let I be a small set and let $F_i \in D^{[a,b]}(\mathbf{k}_{\mathcal{T}})$. Then*

$$(3.6) \quad \bigoplus_i R\rho_{\text{sal}*} F_i \xrightarrow{\sim} R\rho_{\text{sal}*} \left(\bigoplus_i F_i \right).$$

Proof. It is enough to prove that the morphism in (3.6) induces an isomorphism for all $U \in \text{Op}_{M_{\text{sa}}}$:

$$R\Gamma(U; \bigoplus_i R\rho_{\text{sal}*} F_i) \xrightarrow{\sim} R\Gamma(U; R\rho_{\text{sal}*} \bigoplus_i F_i).$$

This follows from Lemma 3.3 and Proposition 1.14.

Q.E.D.

Proposition 3.8. *Let \mathcal{J} be the subcategory of $\text{Mod}(\mathbf{k}_{M_{\text{sa}}})$ consisting of sheaves which are Γ -acyclic. The category \mathcal{J} is $\rho_{\text{sal}*}$ -injective (see [KS06, Cor. 13.3.8]).*

Proof. Let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence in $\text{Mod}(\mathbf{k}_{M_{\text{sa}}})$.

- (i) We see easily that if both F' and F belong to \mathcal{J} , then F'' belongs to \mathcal{J} .
- (ii) It remains to prove that if $F' \in \mathcal{J}$, then the sequence $0 \rightarrow \rho_{\text{sal}*} F' \rightarrow F\rho_{\text{sal}*} \rightarrow \rho_{\text{sal}*} F'' \rightarrow 0$ is exact. Let $U \in \text{Op}_{M_{\text{sa}}}$. By Proposition 1.14 and the hypothesis, the sequence $0 \rightarrow \rho_{\text{sal}*} F'(U) \rightarrow \rho_{\text{sal}*} F(U) \rightarrow \rho_{\text{sal}*} F''(U) \rightarrow 0$ is exact.

Q.E.D.

Applying Proposition 3.5, we get:

Corollary 3.9. *The functor $\rho_{\text{sal}*}$ has cohomological dimension $\leq \dim M$.*

Proposition 3.10. *Let I be a small set and let $F_i \in D(\mathbf{k}_{M_{\text{sa}}})$ ($i \in I$). For $U \in \text{Op}_{M_{\text{sa}}}$, we have the natural isomorphism*

$$(3.7) \quad \bigoplus_{i \in I} R\rho_{\text{sal}*} F_i \xrightarrow{\sim} R\rho_{\text{sal}*} \left(\bigoplus_{i \in I} F_i \right) \text{ in } D(\mathbf{k}_{M_{\text{sal}}}).$$

Proof. The same proof as in Proposition 3.6 holds with the functor $R\rho_{\text{sal}*}$ instead of the functor $R\Gamma(U; \bullet)$. We use Corollary 3.9 instead of Lemma 3.4 and we use (3.6) in Lemma 3.6 instead of (3.3).

Q.E.D.

Theorem 3.11. (i) *The functor $R\rho_{\text{sal}*}: D(\mathbf{k}_{M_{\text{sa}}}) \rightarrow D(\mathbf{k}_{M_{\text{sal}}})$ admits a right adjoint $\rho_{\text{sal}}^!: D(\mathbf{k}_{M_{\text{sal}}}) \rightarrow D(\mathbf{k}_{M_{\text{sa}}})$.*

(ii) The functor $\rho_{\text{sal}}^!$ induces a functor $\rho_{\text{sal}}^!: D^+(\mathbf{k}_{M_{\text{sal}}}) \rightarrow D^+(\mathbf{k}_{M_{\text{sa}}})$.

Proof. (i) follows from the Brown representability theorem (see for example [KS06, Cor. 14.2.3]) and Proposition 3.10.

(ii) This follows from the fact that the functor $\rho_{\text{sal}*}$ has cohomological dimension $\leq \dim M$ by Corollary 3.9 and the general well-known result below. Q.E.D.

Lemma 3.12. *Let $\rho_*: \mathcal{C} \rightarrow \mathcal{C}'$ be a left exact functor between two Grothendieck categories. Assume that $R\rho_*: D(\mathcal{C}) \rightarrow D(\mathcal{C}')$ admits a right adjoint $\rho^!: D(\mathcal{C}') \rightarrow D(\mathcal{C})$ and assume moreover that ρ_* has finite cohomological dimension. Then the functor $\rho^!$ sends $D^+(\mathcal{C}')$ to $D^+(\mathcal{C})$.*

Proof. By the hypothesis, we have for $X \in D(\mathcal{C})$ and $Y \in D(\mathcal{C}')$

$$\text{Hom}_{D(\mathcal{C}')} (R\rho_* X, Y) \simeq \text{Hom}_{D(\mathcal{C})} (X, \rho^! Y).$$

Assume that the cohomological dimension of the functor ρ_* is $\leq r$. Let $Y \in D^{\geq 0}(\mathcal{C}')$. Then $\text{Hom}_{D(\mathcal{C}')} (X, \rho^! Y) \simeq 0$ for all $X \in D^{< -r}(\mathcal{C})$. This means that Y belongs to the right orthogonal to $D^{< -r}(\mathcal{C})$ and this implies that $Y \in D^{\geq -r}(\mathcal{C}')$. Q.E.D.

4 Open sets with Lipschitz boundaries

Normal cones and Lipschitz boundaries

In this paragraph \mathbb{R}^n is equipped with coordinates (x', x_n) , $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

Definition 4.1. We say that $U \in \text{Op}_{M_{\text{sa}}}$ has Lipschitz boundary (or simply, “ U is Lipschitz”) if, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-Lipschitz subanalytic homeomorphism $\psi: V \xrightarrow{\sim} W$ with W an open subset of \mathbb{R}^n such that $\psi(V \cap U) = W \cap \{x_n > 0\}$.

Note that the property of being Lipschitz is local and thus the preceding definition extends to subanalytic but not necessarily relatively compact open subsets of M .

Lemma 4.2. *Let $U \in \text{Op}_{M_{\text{sa}}}$. We assume that, for any $x \in \partial U$, there exist an open neighborhood V of x and a bi-analytic isomorphism $\psi: V \xrightarrow{\sim} W$ with W an open subset of \mathbb{R}^n such that $\psi(V \cap U) = W \cap \{(x', x_n); x_n > \varphi(x')\}$ for a Lipschitz subanalytic function φ . Then U is Lipschitz.*

Proof. We define $\psi_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x', x_n) \mapsto (x', x_n - \varphi(x'))$. Then ψ_1 is a bi-Lipschitz subanalytic homeomorphism and we have $(\psi_1 \circ \psi)(V \cap U) = \psi_1(W) \cap \{x_n > 0\}$. Hence U is Lipschitz. Q.E.D.

Lemma 4.3. *Let \mathbb{V} be a vector space and let γ be a proper closed convex cone with non empty interior. Let $U \in \text{Op}_{\mathbb{V}_{\text{sa}}}$. Then the open set $U + \gamma$ has Lipschitz boundary.*

Proof. Let $p \in \partial(U + \gamma)$. We identify \mathbb{V} with \mathbb{R}^n so that p is the origin and γ contains the cone $\gamma_0 = \{(x', x_n); x_n > \|x'\|\}$. We have in particular

$$(4.1) \quad \gamma_0 \subset (U + \gamma) \subset (\mathbb{R}^n \setminus (-\gamma_0)).$$

For $x' \in \mathbb{R}^{n-1}$ we set $l_{x'} = (U + \gamma) \cap (\{x'\} \times \mathbb{R})$. Then $l_{x'} = l_{x'} + [0, +\infty[$. By (4.1) we also have $l_{x'} \neq \emptyset$ and $l_{x'} \neq \mathbb{R}$. Hence we can write $l_{x'} =]\varphi(x'), +\infty[$, for a well-defined function $\varphi: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$.

Let us prove that φ is Lipschitz. Let $x' \in \mathbb{R}^{n-1}$ and let us set $q = (x', \varphi(x')) \in \partial(U + \gamma)$. We have the similar inclusion as (4.1), $(q + \gamma_0) \subset (U + \gamma) \subset (\mathbb{R}^n \setminus (q - \gamma_0))$. Hence $\partial(U + \gamma) \subset (\mathbb{R}^n \setminus ((q + \gamma_0) \cup (q - \gamma_0)))$. For any $y' \in \mathbb{R}^{n-1}$ we have $(y', \varphi(y')) \in \partial(U + \gamma)$ and the last inclusion translates into $|\varphi(y') - \varphi(x')| \leq \|y' - x'\|$. Hence φ is Lipschitz and $U + \gamma$ is Lipschitz by Lemma 4.2. Q.E.D.

For two subsets A and B of M , we refer to [KS90, Def 4.1.1] for the definition of the normal cone $C(A, B)$.

Definition 4.4. (See [KS90, § 5.3].) Let S be a subset of M . The strict normal cone $N_x(S)$ and the conormal cone $N_x^*(S)$ of S at $x \in M$ as well as the strict normal cone $N(S)$ and the conormal cone $N^*(S)$ of S are given by

$$\begin{aligned} N_x(S) &= T_x M \setminus C(M \setminus S, S), \text{ an open cone in } T_x M, \\ N_x^*(S) &= N_x(S)^\circ, \\ N(S) &= \bigcup_{x \in M} N_x(S), \text{ an open convex cone in } TM, \\ N^*(S) &= \bigcup_{x \in M} N_x^*(S). \end{aligned}$$

By loc. cit. Prop. 5.3.7, we have:

Lemma 4.5. *Let U be an open subset of M and let $x \in \partial U$. Then the conditions below are equivalent:*

- (i) $N_x(U)$ is non empty,
- (ii) $N_y(U)$ is non empty for all y in a neighborhood of x ,
- (iii) $N_x^*(U)$ is contained in a closed convex proper cone with non empty interior in T_x^*M ,
- (iv) there exists a local chart in a neighborhood of x such that identifying M with an open subset of \mathbb{V} , there exists a closed convex proper cone with non empty interior γ in \mathbb{V} such that U is γ -open in an open neighborhood W of x , that is,

$$W \cap ((U \cap W) + \gamma) \subset U.$$

Definition 4.6. We shall say that an open subset U of M satisfies a cone condition if for any $x \in \partial U$, $N_x(U)$ is non empty.

By Lemmas 4.3 and 4.5 we have:

Proposition 4.7. *Let $U \in \text{Op}_{M_{\text{sa}}}$. If U satisfies a cone condition, then U is Lipschitz.*

A vanishing theorem

The next theorem is a key result for this paper and its proof is due to A. Parusinski [Pa12].

Theorem 4.8. (A. Parusinski) *Let $V \in \text{Op}_{M_{\text{sa}}}$. Then there exists a finite covering $V = \bigcup_{j \in J} V_j$ with $V_j \in \text{Op}_{M_{\text{sa}}}$ such that the family $\{V_j\}_{j \in J}$ is a covering of V in M_{sal} and moreover $H^k(V_j; \mathbf{k}_M) \simeq 0$ for all $k > 0$ and all $j \in J$.*

Recall that one denotes by $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$ the natural morphism of sites.

Lemma 4.9. *We have $R\rho_{\text{sal}*}\mathbf{k}_{M_{\text{sa}}} \simeq \mathbf{k}_{M_{\text{sal}}}$.*

Proof. The sheaf $H^k(R\rho_{\text{sal}*}\mathbf{k}_{M_{\text{sa}}})$ is the sheaf associated with the presheaf $U \mapsto H^k(U; \mathbf{k}_{M_{\text{sa}}})$. This sheaf is zero for $k > 0$ by Theorem 4.8. Q.E.D.

Lemma 4.10. *Let $M = \mathbb{R}^n$ and let U be the open subset $U =]0, +\infty[\times \mathbb{R}^{n-1}$. Then we have $R\rho_{\text{sal}*}\mathbf{k}_U \simeq \mathbf{k}_U$.*

Proof. (i) The sheaf $H^k(\mathrm{R}\rho_{\mathrm{sal}*}\mathbf{k}_U)$ is the sheaf associated with the presheaf $V \mapsto H^k(V; \mathbf{k}_U)$. Hence it is enough to show that any $V \in \mathrm{Op}_{M_{\mathrm{sa}}}$ admits a finite covering $V = \bigcup_{j \in J} V_j$ in M_{sal} such that $H^k(V_j; \mathbf{k}_U) \simeq 0$ for all $k > 0$. We assume that the distance d is a subanalytic function. We set

$$V' = \{x \in V; d(x, V \setminus U) < d(x, M \setminus V)\}.$$

This is a subanalytic open subset of V . We remark that $V \cap (]-\infty, 0] \times \mathbb{R}^{n-1}) \subset V'$, hence $V = V' \cup (V \cap U)$.

(ii) Let us prove that $\{V', V \cap U\}$ is a covering of V in M_{sal} . We assume that $d(x, M \setminus V) \geq 3d(x, M \setminus (V \cap U))$, for a given $x \in V$, and we prove that $d(x, M \setminus V') \geq d(x, M \setminus V)/3$.

We denote by $B(x, r)$ the open ball with center x and radius r . We set $R = d(x, M \setminus V)/3$. Hence $B(x, 3R) \subset V$. By hypothesis we can choose $y \in M \setminus (V \cap U)$ such that $d(x, y) \leq R$. In particular $y \in B(x, 3R) \subset V$, hence $y \in V \setminus U$. Let us prove that $B(x, R) \subset V'$. For any $z \in B(x, R)$, we have

$$d(z, V \setminus U) \leq d(z, y) \leq d(z, x) + d(x, y) < 2R.$$

We also have $d(z, M \setminus V) \geq d(x, M \setminus V) - d(z, x) = 3R - d(z, x) > 2R$. Hence $z \in V'$ by definition of V' . Since this holds for any $z \in B(x, R)$, we have $d(x, M \setminus V') \geq R$, as claimed.

(iii) Let us prove that $\mathrm{R}\Gamma(V'; \mathbf{k}_U) \simeq 0$. We take coordinates (x_1, x') on $M = \mathbb{R}^n$. For $x = (x_1, x')$ with $x_1 \geq 0$, we have $d(x, V \setminus U) \geq d(x, M \setminus U) = x_1$. If $(x_1, x') \in V'$ we obtain $d(x, M \setminus V) > x_1$, hence $\overline{B(x, x_1)} \subset V$. We deduce that $V' \cap \overline{U} = \{x = (x_1, x') \in V; x_1 \geq 0 \text{ and } \overline{B(x, x_1)} \subset V\}$. It follows easily that, if $(x_1, x') \in V' \cap \overline{U}$, then $(y_1, x') \in V' \cap \overline{U}$, for all $y_1 \in [0, x_1]$. Let $q: \mathbb{R}^n \rightarrow \{0\} \times \mathbb{R}^{n-1}$ be the projection. We deduce:

(a) q maps $V' \cap \overline{U}$ onto $V \cap \partial U$,

(b) $q^{-1}(x) \cap V' \cap U$ is an open interval, for any $x = (0, x') \in V \cap \partial U$.

For any $a < 0 < b$ we have $\mathrm{R}\Gamma(]a, b[; \mathbf{k}_{]0, b[}) \simeq 0$. Hence (a) and (b) give $\mathrm{R}q_*\mathrm{R}\Gamma_{V'}\mathbf{k}_U \simeq 0$, by the base change formula, and we obtain $\mathrm{R}\Gamma(V'; \mathbf{k}_U) \simeq \mathrm{R}\Gamma(\mathbb{R}^{n-1}; \mathrm{R}q_*\mathrm{R}\Gamma_{V'}\mathbf{k}_U) \simeq 0$.

(iv) By Theorem 4.8 we can choose a finite covering of $V \cap U$ in M_{sal} , say $\{W_j\}_{j \in J}$, such that $H^k(W_j; \mathbf{k}_U) \simeq 0$ for all $k > 0$. By (ii) the family $\{V', \{W_j\}_{j \in J}\}$ is a covering of V in M_{sal} . By (iii) this covering satisfies the required condition in (i), which proves the result. Q.E.D.

Theorem 4.11. *Let $U \in \text{Op}_{M_{\text{sa}}}$ and assume that U has a Lipschitz boundary.*

- (i) $R\rho_{\text{sal}*}\mathbf{k}_{UM_{\text{sa}}} \simeq \rho_{\text{sal}*}\mathbf{k}_{UM_{\text{sa}}} \simeq \mathbf{k}_{UM_{\text{sal}}}$ *is concentrated in degree zero.*
- (ii) *For $F \in \text{D}^b(\mathbf{k}_{M_{\text{sal}}})$, one has $R\Gamma(U; \rho_{\text{sal}}^!F) \simeq R\Gamma(U; F)$.*
- (iii) *Let $F \in \text{Mod}(\mathbf{k}_{M_{\text{sal}}})$ and assume that F is Γ -acyclic. Then $R\Gamma(U; \rho_{\text{sal}}^!F)$ is concentrated in degree 0 and is isomorphic to $F(U)$.*

Note that the result in (i) is local and it is not necessary to assume here that U is relatively compact.

Proof. (i) This is a local problem. Hence, by Remark 1.10 and by the definition of “Lipschitz boundary” the first isomorphism follows from Lemma 4.10. The second one is given in Proposition 1.12.

(ii) follows from (i) and the adjunction between $R\rho_{\text{sal}*}$ and $\rho_{\text{sal}}^!$.

(iii) follows from (ii).

Q.E.D.

Example 4.12. ¹ Let $M = \mathbb{R}^2$ endowed with coordinates $x = (x_1, x_2)$. Let $R > 0$ and denote by B_R the open Euclidian ball with center 0 and radius R . Consider the subanalytic sets:

$$U_1 = \{x \in B_R; x_1 > 0, x_2 < x_1^2\}, \quad U_2 = \{x \in B_R; x_1 > 0, x_2 > -x_1^2\}, \\ U_{12} = U_1 \cap U_2, \quad U = U_1 \cup U_2 = \{x \in B_R; x_1 > 0\}.$$

Note that $\{U_1, U_2\}$ is a covering of U in M_{sa} but not in M_{sal} . Denote for short by $\rho: M_{\text{sa}} \rightarrow M_{\text{sal}}$ the morphism ρ_{sal} . We have the distinguished triangle in $\text{D}^b(\mathbf{k}_{M_{\text{sal}}})$:

$$(4.2) \quad R\rho_*\mathbf{k}_{U_{12}} \rightarrow R\rho_*\mathbf{k}_{U_1} \oplus R\rho_*\mathbf{k}_{U_2} \rightarrow R\rho_*\mathbf{k}_U \xrightarrow{+1}.$$

Since U_1, U_2 and U are Lipschitz, $R\rho_*\mathbf{k}_V$ is concentrated in degree 0 for $V = U_1, U_2, U$. It follows that $R\rho_*\mathbf{k}_{U_{12}}$ is concentrated in degrees 0 and 1. Hence, we have the distinguished triangle

$$(4.3) \quad \rho_*\mathbf{k}_{U_{12}} \rightarrow R\rho_*\mathbf{k}_{U_{12}} \rightarrow R^1\rho_*\mathbf{k}_{U_{12}}[-1] \xrightarrow{+1}.$$

¹Examples 4.12, 4.13 and 4.14 emerged from discussions with G. Lebeau

Let us prove that $R^1\rho_*\mathbf{k}_{U_{12}}$ is isomorphic to the sheaf N introduced in (1.13). We easily see that there exists a natural morphism $\mathbf{k}_U \rightarrow N$ which is surjective. Hence we have to prove that the sequence

$$\mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \rightarrow \mathbf{k}_U \rightarrow N$$

is exact. This reduces to the following assertion: if $V \in \text{Op}_{M_{\text{sa}}}$ satisfies $V \subset U$ and $N(V) = 0$, then $\{V \cap U_1, V \cap U_2\}$ is a linear covering of V . We prove this claim now.

Let $V \subset U$ be such that $N(V) = 0$. By the definition of N , there exists $A > 0$ such that $U_{A,\varepsilon} \not\subset V$ for all $\varepsilon > 0$, where $U_{A,\varepsilon}$ is defined in (1.11). Hence there exists a sequence $\{(x_{1,n}, x_{2,n})\}_{n \in \mathbb{N}}$ such that $x_{1,n} > 0$, $x_{1,n} \rightarrow 0$ when $n \rightarrow \infty$, $|x_{2,n}| < Ax_{1,n}^2$ and $(x_{1,n}, x_{2,n}) \notin V$, for all $n \in \mathbb{N}$. We define $f(x) = d((x, 0), M \setminus V)$, for $x \in \mathbb{R}$. Then f is a continuous subanalytic function and $f(x_{1,n}) < Ax_{1,n}^2$, for all $n \in \mathbb{N}$. It follows, by the same argument as in Example 1.17 (ii), that there exists $x_0 > 0$ such that $f(x) \leq Ax^2$ for all $x \in]0, x_0[$. We deduce, for any $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 \in]0, x_0[$,

$$(4.4) \quad d((x_1, x_2), M \setminus V) \leq |x_2| + d((x_1, 0), M \setminus V) \leq |x_2| + Ax_1^2.$$

On the other hand we can find $B > 0$ such that, for any $(x_1, x_2) \in U$,

$$(4.5) \quad \max\{d((x_1, x_2), M \setminus U_1), d((x_1, x_2), M \setminus U_2)\} \geq |x_2| + Bx_1^2.$$

We deduce easily from (4.4) and (4.5) that $\{V \cap U_1, V \cap U_2\}$ is a linear covering of V .

Example 4.13. We keep the notations of Example 4.12. Let F be a sheaf on M_{sal} which is Γ -acyclic. It follows from Theorem 4.11 that $\text{R}\Gamma(V; \rho^!F)$ is concentrated in degree 0 for $V = U_1, U_2, U$. Applying the functor $\text{R}\Gamma(\cdot; \rho^!F)$ to the distinguished triangle (4.2) we get the exact sequence

$$0 \rightarrow F(U) \rightarrow F(U_1) \oplus F(U_2) \rightarrow H^0\text{R}\Gamma(U_{12}; \rho^!F) \rightarrow 0.$$

In other words, the object $\text{R}\Gamma(U_{12}; \rho^!F) \simeq \text{RHom}_{\mathbf{k}_{M_{\text{sa}}}}(\text{R}\rho_*\mathbf{k}_{U_{12}}, F)$ is concentrated in degree 0 and is isomorphic to the cokernel of the map $F(U) \rightarrow F(U_1) \oplus F(U_2)$. Using the distinguished triangle (4.3), we get the long exact sequence

$$\begin{aligned} 0 \rightarrow H^1\text{RHom}_{\mathbf{k}_{M_{\text{sa}}}}(N, F) \rightarrow H^0\text{RHom}_{\mathbf{k}_{M_{\text{sa}}}}(\text{R}\rho_*\mathbf{k}_{U_{12}}, F) \rightarrow \\ F(U_{12}) \rightarrow H^2\text{RHom}_{\mathbf{k}_{M_{\text{sa}}}}(N, F) \rightarrow 0. \end{aligned}$$

In particular, $H^0\mathrm{RHom}_{\mathbf{k}_{M_{\mathrm{sa}}}}(N, F) \simeq 0$ and if the map $F(U_1) \oplus F(U_2) \rightarrow F(U_{12})$ is not surjective, then $H^2\mathrm{RHom}_{\mathbf{k}_{M_{\mathrm{sa}}}}(N, F) \neq 0$.

Example 4.14. Let $M = \mathbb{R}^2$, $x = (x_1, x_2)$, B_R and $\rho: M_{\mathrm{sa}} \rightarrow M_{\mathrm{sal}}$ be as in Example 4.13. Consider the subanalytic sets:

$$\begin{aligned} U_1 &= \{x \in B_R; x_1 > -|x_2|\}, & U_2 &= \{x \in B_R; x_1 < |x_2|\}, \\ U_{12} &= U_1 \cap U_2, & U &= U_1 \cup U_2 = \{x \in B_R; x \neq 0\}. \end{aligned}$$

Then U_1 and U_2 are Lipschitz and U_{12} is a disjoint union of two Lipschitz open sets. Therefore, $\mathrm{R}\rho_*\mathbf{k}_{U_{12}}$ is concentrated in degree 0 and it follows from the distinguished triangle (4.3) that $\mathrm{R}\rho_*\mathbf{k}_U$ is also concentrated in degree 0.

5 Construction of sheaves I

In this section, we shall construct sheaves on the site M_{sal} .

On the site M_{sa} , the sheaves $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty, \mathrm{temp}}$ and $\mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{temp}}$ below have been constructed in [KS96, KS01]. By using the linear topology M_{sal} we shall construct sheaves on M_{sal} associated with more precise growth conditions.

Let us choose a distance d on M such that, for any $x \in M$ and any local chart $(U, \varphi: U \hookrightarrow \mathbb{R}^n)$ around x , there exists a neighborhood of x over which d is Lipschitz equivalent to the pull-back of the Euclidean distance by φ .

The sheaves $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty, \mathrm{temp}}$ and $\mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{temp}}$

For the reader's convenience, let us recall first some definitions of [KS96, KS01]. As usual, we denote by \mathcal{C}_M^∞ (resp. \mathcal{C}_M^ω) the sheaf of complex functions of class \mathcal{C}^∞ (resp. real analytic), by $\mathcal{D}b_M$ (resp. \mathcal{B}_M) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions) and by \mathcal{D}_M the sheaf of analytic finite-order differential operators. We also use the notation $\mathcal{A}_M = \mathcal{C}_M^\omega$.

Definition 5.1. Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $f \in \mathcal{C}_M^\infty(U)$. One says that f has *polynomial growth* at $p \in M$ if it satisfies the following condition. For a local coordinate system (x_1, \dots, x_n) around p , there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$(5.1) \quad \sup_{x \in K \cap U} (\mathrm{dist}(x, K \setminus U))^N |f(x)| < \infty.$$

It is obvious that f has polynomial growth at any point of U . We say that f is temperate at p if all its derivatives have polynomial growth at p . We say that f is temperate if it is temperate at any point.

For $U \in \text{Op}_{M_{\text{sa}}}$, we shall denote by $\mathcal{C}_M^{\infty, \text{temp}}(U)$ the subspace of $\mathcal{C}_M^\infty(U)$ consisting of tempered functions and by $\mathcal{D}b_M^{\text{temp}}(U)$ the space of tempered distributions on U , defined by the exact sequence

$$0 \rightarrow \Gamma_{M \setminus U}(M; \mathcal{D}b_M) \rightarrow \Gamma(M; \mathcal{D}b_M) \rightarrow \mathcal{D}b_M^{\text{temp}}(U) \rightarrow 0.$$

It follows from the work of Lojasiewicz that $U \mapsto \mathcal{C}_M^{\infty, \text{temp}}(U)$ and $U \mapsto \mathcal{D}b_M^{\text{temp}}(U)$ are sheaves on M_{sa} . We denote by $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}$ and $\mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}$ these sheaves on M_{sa} . Note that $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}$ and $\mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}$ are Γ -acyclic and the sheaf $\mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}$ is flabby (see Definition 2.6).

Recall that one denotes by $\rho_{\text{sa}}: M \rightarrow M_{\text{sa}}$ and $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$ the natural morphisms of sites. We set:

$$\mathcal{D}_{M_{\text{sa}}} := \rho_{\text{sa}!} \mathcal{D}_M.$$

Hence, $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}$ and $\mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}$ are $\mathcal{D}_{M_{\text{sa}}}$ -modules.

Since $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}$, $\mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}$ and $\mathcal{D}_{M_{\text{sa}}}$ are Γ -acyclic on M_{sa} , we have by Proposition 3.8

$$\text{R}\rho_{\text{sal}*} \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}, \text{R}\rho_{\text{sal}*} \mathcal{D}b_{M_{\text{sa}}}^{\text{temp}} \text{ and } \text{R}\rho_{\text{sal}*} \mathcal{D}_{M_{\text{sa}}} \text{ are concentrated in degree 0.}$$

In the sequel, we shall use the following notations. We set

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{temp}} := \rho_{\text{sal}*} \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}, \quad \mathcal{D}b_{M_{\text{sal}}}^{\text{temp}} := \rho_{\text{sal}*} \mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}, \quad \mathcal{D}_{M_{\text{sal}}} := \rho_{\text{sal}*} \mathcal{D}_{M_{\text{sa}}}.$$

Remark 5.2. The sheaves $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{temp}}$ and $\mathcal{D}b_{M_{\text{sa}}}^{\text{temp}}$ are respectively denoted by $\mathcal{C}_M^{\infty, t}$ and $\mathcal{D}b_M^t$ in [KS01].

Sheaves with temperate growth of a given order

Definition 5.3. Let $U \in \text{Op}_{M_{\text{sa}}}$, let $f \in \mathcal{C}_M^\infty(U)$ and let $s \in \mathbb{R}_{\geq 0}$. We say that f has *growth of order* $\leq s$ at $p \in M$ if it satisfies the following condition. For a local coordinate system (x_1, \dots, x_n) around p , there exists a sufficiently small compact neighborhood K of p such that

$$(5.2) \quad \sup_{x \in K \cap U} (\text{dist}(x, K \setminus U))^s |f(x)| < \infty.$$

It is obvious that f has growth of order $\leq s$ at any point of U . We say that f is temperate of order s at p if, for each $m \in \mathbb{N}$, all its derivatives of order $\leq m$ have polynomial growth of order $\leq s + m$ at p . We say that f is temperate of order s if it is temperate of order s at any point.

For $U \in \text{Op}_{M_{\text{sa}}}$, we denote by $\mathcal{C}_M^{\infty,s}(U)$ the subspace of $\mathcal{C}_M^\infty(U)$ consisting of functions tempered of order s and we denote by $\mathcal{C}_{M_{\text{sal}}}^{\infty,s}$ the presheaf on M_{sal} so obtained.

The next result is clear.

- Proposition 5.4.** (i) *The presheaves $\mathcal{C}_{M_{\text{sal}}}^{\infty,s}$ are sheaves on M_{sal} ,*
(ii) *the sheaf $\mathcal{C}_{M_{\text{sal}}}^{\infty,0}$ is a sheaf of rings,*
(iii) *for $s \geq 0$, $\mathcal{C}_{M_{\text{sal}}}^{\infty,s}$ is a $\mathcal{C}_{M_{\text{sal}}}^{\infty,0}$ -module and there are natural morphisms*

$$\mathcal{C}_{M_{\text{sal}}}^{\infty,s} \otimes_{\mathcal{C}_{M_{\text{sal}}}^{\infty,0}} \mathcal{C}_{M_{\text{sal}}}^{\infty,s'} \rightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty,s+s'}.$$

We also introduce the sheaf

$$\mathcal{C}_{M_{\text{sal}}}^{\infty,tp} := \varinjlim_s \mathcal{C}_{M_{\text{sal}}}^{\infty,s}.$$

(Of course, the limit is taken in the category of sheaves on M_{sal} .) Then, for $0 \leq s \leq s'$, there are natural monomorphisms of sheaves on M_{sal} :

$$(5.3) \quad \mathcal{C}_{M_{\text{sal}}}^{\infty,0} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty,s} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty,s'} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty,tp} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty,\text{temp}}.$$

A refined cutoff lemma

Lemma 5.5 below will play an important role in this paper and is an immediate corollary of a result of Hörmander [Ho83, Cor.1.4.11]. Note that Hörmander's result was already used in [KS96, Prop. 10.2]

Lemma 5.5. *Let Z_1 and Z_2 be two closed subsets of $M := \mathbb{R}^n$. Assume that there exists $C > 0$ such that*

$$(5.4) \quad d(x, Z_1 \cap Z_2) \leq C(d(x, Z_1) + d(x, Z_2)) \text{ for any } x \in M.$$

Then there exists $\psi \in \mathcal{C}_M^{\infty,0}(M \setminus (Z_1 \cap Z_2))$ such that $\psi = 0$ on a neighborhood of $Z_1 \setminus Z_2$ and $\psi = 1$ on a neighborhood of $Z_2 \setminus Z_1$.

Let $U_1, U_2 \in \text{Op}_{M_{\text{sa}}}$ and let $U = U_1 \cup U_2$. For $a > 0$ we set

$$\begin{aligned} Z_1^a &= \{x \in M; d(x, M \setminus U_2) \leq a d(x, M \setminus U_1)\}, \\ Z_2^a &= \{x \in M; d(x, M \setminus U_1) \leq a d(x, M \setminus U_2)\}. \end{aligned}$$

Note that $Z_i^a \cap U \subset U_i$, for $i = 1, 2$ and $a < 1$, that $Z_i^a \subset Z_i^b$ for $a \leq b$, that $M \setminus U \subset Z_1^a \cap Z_2^a$ for all $a > 0$, with equality if $a < 1$, and that $Z_1^1 \cup Z_2^1 = M$.

Lemma 5.6. *We have, for any $z \in M$,*

- (i) *if $z \in Z_1^1$, then $d(z, M \setminus U_1) \leq 3 d(z, Z_2^{1/2})$,*
- (ii) *$\max_{i=1,2} \{d(z, M \setminus U_i)\} \leq 3 \max_{i=1,2} \{d(z, Z_i^{1/2})\}$,*
- (iii) *if $z \in Z_1^{1/4}$, then $d(z, M \setminus Z_1^{1/2}) \geq (1/8)d(z, M \setminus U_1)$.*

Proof. (i) We choose $x \in Z_2^{1/2}$. Then we have

$$\begin{aligned} d(z, M \setminus U_1) - d(z, x) &\leq d(x, M \setminus U_1) \\ &\leq (1/2)d(x, M \setminus U_2) \\ &\leq (1/2)(d(z, x) + d(z, M \setminus U_2)) \\ &\leq (1/2)(d(z, x) + d(z, M \setminus U_1)), \end{aligned}$$

where the first and third lines follow easily from the triangular inequality, the second and last lines follow from the definitions of $Z_2^{1/2}$ and Z_1^1 . We deduce $d(z, M \setminus U_1) \leq 3d(z, x)$. We can take x such that $d(z, x) = d(z, Z_2^{1/2})$ and (i) follows.

(ii) We remark that (i) admits a symmetric statement where we exchange the subscripts 1 and 2. We also have $Z_1^1 \cup Z_2^1 = M$. By symmetry we may assume that $z \in Z_1^1$, that is, $\max_{i=1,2} \{d(z, M \setminus U_i)\} = d(z, M \setminus U_1)$. Then (ii) follows from (i).

(iii) Let $x \in M$ be such that $d(z, x) \leq (1/8)d(z, M \setminus U_1)$. Then we have

$$(5.5) \quad d(x, M \setminus U_1) \geq d(z, M \setminus U_1) - d(z, x) \geq 7d(z, x)$$

and

$$\begin{aligned}
d(x, M \setminus U_2) &\leq d(z, x) + d(z, M \setminus U_2) \\
&\leq d(z, x) + (1/4)d(z, M \setminus U_1) \\
&\leq d(z, x) + (1/4)(d(z, x) + d(x, M \setminus U_1)) \\
&\leq (5/4)d(z, x) + (1/4)d(x, M \setminus U_1) \\
&\leq (5/28 + 1/4)d(x, M \setminus U_1) \\
&\leq (1/2)d(x, M \setminus U_1),
\end{aligned}$$

where the first and third lines are triangular inequalities, the second line follows from “ $z \in Z_1^{1/4}$ ” and the fifth from (5.5). Hence we have proved that, if $d(z, x) \leq (1/8)d(z, M \setminus U_1)$, then $x \in Z_1^{1/2}$. This means that $Z_1^{1/2}$ contains the ball centered at z with radius $(1/8)d(z, M \setminus U_1)$ and this gives (iii).
Q.E.D.

Lemma 5.7. *Assume that $\{U_1, U_2\}$ is a linear covering of U . Then there exists $\psi \in \mathcal{C}_M^{\infty, 0}(U)$ such that $\psi = 0$ on a neighborhood of $Z_1^{1/2} \setminus Z_2^{1/2}$ and $\psi = 1$ on a neighborhood of $Z_2^{1/2} \setminus Z_1^{1/2}$.*

Proof. Let $A > 0$ be such that $d(z, M \setminus U) \leq A \max_{i=1,2} \{d(z, M \setminus U_i)\}$, for all $z \in M$. By (ii) of Lemma 5.6 the inequality (5.4) holds with $C = 3A$ and Lemma 5.5 gives the result.
Q.E.D.

Lemma 5.8. *The pair $\{\text{Int} Z_1^{1/2}, U_1 \cap U_2\}$ is a linear covering of U_1 and $\{\text{Int} Z_2^{1/2}, U_1 \cap U_2\}$ is a linear covering of U_2 .*

Proof. By symmetry we only have to prove the first assertion. Let $x \in U_1$. If $d(x, M \setminus U_2) \geq (1/4)d(x, M \setminus U_1)$, then $d(x, M \setminus (U_1 \cap U_2)) \geq (1/4)d(x, M \setminus U_1)$. In the other case we have $x \in Z_1^{1/4}$. By (iii) of Lemma 5.6 we have $d(x, M \setminus Z_1^{1/2}) \geq (1/8)d(x, M \setminus U_1)$. Since $x \in U_1$, we have $r := (1/8)d(x, M \setminus U_1) > 0$. Hence $Z_1^{1/2}$ contains the open ball $B(x, r)$. But then $\text{Int} Z_1^{1/2}$ contains the same ball and we have $d(x, M \setminus \text{Int} Z_1^{1/2}) \geq (1/8)d(x, M \setminus U_1)$. In both cases

$$d(x, M \setminus U_1) \leq 8 \max\{d(x, M \setminus (U_1 \cap U_2)), d(x, M \setminus \text{Int} Z_1^{1/2})\},$$

as required.

Q.E.D.

Proposition 5.9. *Let \mathcal{F} be a sheaf of $\mathcal{C}_{M_{\text{sal}}}^{\infty, 0}$ -modules on M_{sal} . Then \mathcal{F} is Γ -acyclic.*

Proof. By Proposition 2.4, it is enough to prove that for any $\{U_1, U_2\}$ which is a covering of $U_1 \cup U_2$, the sequence $0 \rightarrow \mathcal{F}(U_1 \cup U_2) \rightarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \rightarrow \mathcal{F}(U_1 \cap U_2) \rightarrow 0$ is exact. This follows from Lemma 5.7, similarly as in the proof of [KS96, Prop. 10.2]. We use the notations $Z_1^{1/2}, Z_2^{1/2}, \psi$ of Lemma 5.7. By Lemma 5.8, $\{\text{Int}Z_1^{1/2}, U_1 \cap U_2\}$ is a linear covering of U_1 and $\{\text{Int}Z_2^{1/2}, U_1 \cap U_2\}$ is a linear covering of U_2 . Hence, for a given $s \in \Gamma(U_1 \cap U_2; \mathcal{F})$ we can define $s_1 \in \Gamma(U_1; \mathcal{F})$ and $s_2 \in \Gamma(U_2; \mathcal{F})$ by

$$s_1|_{U_1 \cap U_2} = \psi \cdot s, \quad s_1|_{\text{Int}Z_1^{1/2}} = 0 \quad \text{and} \quad s_2|_{U_1 \cap U_2} = (1 - \psi) \cdot s, \quad s_2|_{\text{Int}Z_2^{1/2}} = 0.$$

Then $s_1|_{U_1 \cap U_2} + s_2|_{U_1 \cap U_2} = s$.

Q.E.D.

Corollary 5.10. (i) *The sheaves $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{temp}}$ and $\mathcal{D}b_{M_{\text{sal}}}^{\text{temp}}$ are Γ -acyclic.*
(ii) *The sheaves $\mathcal{C}_{M_{\text{sal}}}^{\infty, s}$ ($s \in \mathbb{R}_{\geq 0}$) and $\mathcal{C}_{M_{\text{sal}}}^{\infty, tp}$ are Γ -acyclic.*

Gevrey sheaves

The definition below of the sheaves $\mathcal{G}^{(s)}$ and $\mathcal{G}^{\{s\}}$ is inspired by the definition of the sheaves of C^∞ -functions of Gevrey classes, but is completely different from the classical one. Here we are interested in the growth of the function at the boundary contrarily to the classical setting where one is interested in the Taylor expansion of the function. As usual, there are two kinds of regularity which can be interesting: regularity at the interior or at the boundary. Since we shall soon consider the Dolbeault complexes of our new sheaves, the interior regularity is irrelevant and we are only interested in the regularity (or better, “growth”) at the boundary.

We refer to [Ko73, Ko77] for an exposition on classical Gevrey functions or distributions and their link with Sato’s theory of boundary values of holomorphic functions. Note that there is also a recent study by [HM11] of these sheaves using the tools of subanalytic geometry.

Definition 5.11. Let $U \in \text{Op}_{M_{\text{sa}}}$, let $f \in \mathcal{C}_M^\infty(U)$ and let $(s, h) \in]1, +\infty[\times]0, +\infty[$. We say that f has *0-exponential growth of type (s, h)* at $p \in M$ if it satisfies the following condition. For a local coordinate system (x_1, \dots, x_n) around p , there exists a sufficiently small compact neighborhood K of p such that

$$(5.6) \quad \sup_{x \in K \cap U} \left(\exp(-h \cdot \text{dist}(x, K \setminus U)^{1-s}) \right) |f(x)| < \infty.$$

It is obvious that f has 0-exponential growth of type (s, h) at any point of U . We say that f has exponential growth of type (s, h) at p if all its derivatives have 0-exponential growth of type (s, h) at p . We say that f has exponential growth of type (s, h) if it has such a growth at any point.

We denote by $G_M^{s,h}(U)$ the subspace of $\mathcal{C}_M^\infty(U)$ consisting of functions with exponential growth of type (s, h) .

Definition 5.12. For $U \in \text{Op}_{M_{\text{sa}}}$ and $s \in]1, +\infty[$, we set:

$$G_M^{(s)}(U) := \varprojlim_h G_M^{s,h}(U), \quad G_M^{\{s\}}(U) := \varinjlim_h G_M^{s,h}(U).$$

and we denote by $\mathcal{G}_{M_{\text{sal}}}^{(s)}$ and $\mathcal{G}_{M_{\text{sal}}}^{\{s\}}$ the presheaves on M_{sal} so obtained.

We also set $\mathcal{G}_{M_{\text{sal}}}^{(1)} = \mathcal{G}_{M_{\text{sal}}}^{\{1\}} := \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{temp}}$.

Clearly, the presheaves $\mathcal{G}_{M_{\text{sal}}}^{(s)}$ and $\mathcal{G}_{M_{\text{sal}}}^{\{s\}}$ do not depend on the choice of the distance.

- Proposition 5.13.** (i) *The presheaves $\mathcal{G}_{M_{\text{sal}}}^{(s)}$ and $\mathcal{G}_{M_{\text{sal}}}^{\{s\}}$ are sheaves on M_{sal} ,*
(ii) *the presheaves $\mathcal{G}_{M_{\text{sal}}}^{(s)}$ and $\mathcal{G}_{M_{\text{sal}}}^{\{s\}}$ are $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{temp}}$ -modules,*
(iii) *the presheaves $\mathcal{G}_{M_{\text{sal}}}^{(s)}$ and $\mathcal{G}_{M_{\text{sal}}}^{\{s\}}$ are $\mathcal{D}_{M_{\text{sal}}}$ -modules,*
(iv) *the presheaves $\mathcal{G}_{M_{\text{sal}}}^{(s)}$ and $\mathcal{G}_{M_{\text{sal}}}^{\{s\}}$ are Γ -acyclic,*
(v) *we have natural monomorphisms of sheaves on M_{sal} for $1 < s < s'$*

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{temp}} \hookrightarrow \mathcal{G}_{M_{\text{sal}}}^{(s)} \hookrightarrow \mathcal{G}_{M_{\text{sal}}}^{\{s\}} \hookrightarrow \mathcal{G}_{M_{\text{sal}}}^{(s')} \hookrightarrow \mathcal{G}_{M_{\text{sal}}}^{\{s'\}} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^\infty.$$

Proof. (i), (ii), (iii) and (v) are obvious and (iv) follows from (ii) and Proposition 5.9. Q.E.D.

6 Construction of sheaves II

By using the functor $\rho_{\text{sal}}^!$, we will construct new sheaves (in the derived sense) on M_{sa} associated with the sheaves previously constructed on M_{sal} . Recall that one denotes by $\rho_{\text{sa}}: M \rightarrow M_{\text{sa}}$ and $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$ the natural morphisms of sites.

- Theorem 6.1.** (i) *The functor $\rho_{\text{sal}*}: \text{Mod}(\mathcal{D}_{M_{\text{sa}}}) \rightarrow \text{Mod}(\mathcal{D}_{M_{\text{sal}}})$ has finite cohomological dimension.*
- (ii) *The functor $R\rho_{\text{sal}*}: D(\mathcal{D}_{M_{\text{sa}}}) \rightarrow D(\mathcal{D}_{M_{\text{sal}}})$ commutes with small direct sums.*
- (iii) *The functor $R\rho_{\text{sal}*}$ in (ii) admits a right adjoint $\rho_{\text{sal}}^!: D(\mathcal{D}_{M_{\text{sal}}}) \rightarrow D(\mathcal{D}_{M_{\text{sa}}})$.*
- (iv) *The functor $\rho_{\text{sal}}^!$ induces a functor $\rho_{\text{sal}}^!: D^+(\mathcal{D}_{M_{\text{sal}}}) \rightarrow D^+(\mathcal{D}_{M_{\text{sa}}})$.*

Proof. Consider the quasi-commutative diagram of categories

$$\begin{array}{ccc} \text{Mod}(\mathcal{D}_{M_{\text{sa}}}) & \xrightarrow{\rho_{\text{sal}*}} & \text{Mod}(\mathcal{D}_{M_{\text{sal}}}) \\ \text{for} \downarrow & & \downarrow \text{for} \\ \text{Mod}(\mathbb{C}_{M_{\text{sa}}}) & \xrightarrow{\rho_{\text{sal}*}} & \text{Mod}(\mathbb{C}_{M_{\text{sal}}}). \end{array}$$

The functor $\text{for}: \text{Mod}(\mathcal{D}_{M_{\text{sa}}}) \rightarrow \text{Mod}(\mathbb{C}_{M_{\text{sa}}})$ is exact and sends injective objects to injective objects, and similarly with M_{sal} instead of M_{sa} . It follows that the diagram below commutes:

$$\begin{array}{ccc} D(\mathcal{D}_{M_{\text{sa}}}) & \xrightarrow{R\rho_{\text{sal}*}} & D(\mathcal{D}_{M_{\text{sal}}}) \\ \text{for} \downarrow & & \downarrow \text{for} \\ D(\mathbb{C}_{M_{\text{sa}}}) & \xrightarrow{R\rho_{\text{sal}*}} & D(\mathbb{C}_{M_{\text{sal}}}). \end{array}$$

Moreover, the two functors for in the diagram above are conservative. Then

- (i) follows from Corollary 3.9,
(ii) follows from Proposition 3.10,
(iii)-(iv) follow from the Brown representability theorem, as in the proof of Theorem 3.11. Q.E.D.

Definition 6.2. For $s \geq 0$, we set:

$$\begin{aligned} \mathcal{C}_{M_{\text{sa}}}^{\infty,s} &:= \rho_{\text{sal}}^! \mathcal{C}_{M_{\text{sal}}}^{\infty,s} \in D^+(\mathbb{C}_{M_{\text{sa}}}), \\ \mathcal{C}_{M_{\text{sa}}}^{\infty,tp} &:= \rho_{\text{sal}}^! \mathcal{C}_{M_{\text{sal}}}^{\infty,tp} \in D^+(\mathcal{D}_{M_{\text{sa}}}). \end{aligned}$$

Let us apply Theorem 4.11 and Corollary 5.10. We get that if $U \in \text{Op}_{M_{\text{sa}}}$ is Lipschitz, then

$$\text{R}\Gamma(U; \mathcal{C}_{M_{\text{sa}}}^{\infty,s}) \simeq \mathcal{C}_M^{\infty,s}(U),$$

and similarly with $\mathcal{C}_{M_{\text{sa}}}^{\infty,tp}$.

Definition 6.3. For $s \in [1, +\infty[$, we set:

$$\mathcal{G}_{M_{\text{sa}}}^{(s)} := \rho_{\text{sa}}^! \mathcal{G}_{M_{\text{sal}}}^{(s)} \text{ and } \mathcal{G}_{M_{\text{sa}}}^{\{s\}} := \rho_{\text{sa}}^! \mathcal{G}_{M_{\text{sal}}}^{\{s\}}, \text{ objects of } D^+(\mathcal{D}_{M_{\text{sa}}}).$$

We call $\mathcal{G}_{M_{\text{sa}}}^{(s)}$ the sheaf of Gevrey functions of type (s) and $\mathcal{G}_{M_{\text{sa}}}^{\{s\}}$ the sheaf of Gevrey functions of type $\{s\}$ on M_{sa} .

Applying Theorem 4.11 and Proposition 5.13, we get that if $U \in \text{Op}_{M_{\text{sa}}}$ is Lipschitz, then

$$\text{R}\Gamma(U; \mathcal{G}_{M_{\text{sa}}}^{(s)}) \simeq G_M^{(s)}(U), \quad \text{R}\Gamma(U; \mathcal{G}_{M_{\text{sa}}}^{\{s\}}) \simeq G_M^{\{s\}}(U).$$

Sheaves on complex manifolds

Let X be a complex manifold of complex dimension d_X and denote by $X_{\mathbb{R}}$ the real analytic underlying manifold. Denote by \overline{X} the complex manifold conjugate to X . (The holomorphic functions on \overline{X} are the anti-holomorphic functions on X .) Then $X \times \overline{X}$ is a complexification of $X_{\mathbb{R}}$ and $\mathcal{O}_{\overline{X}}$ is a $\mathcal{D}_{X \times \overline{X}}$ -module which plays the role of the Dolbeault complex.

Recall that the sheaf $\mathcal{O}_{X_{\text{sa}}}^{\text{temp}} := \text{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sa}}^! \mathcal{O}_{\overline{X}_{\text{sa}}}, \mathcal{C}_{X_{\text{sa}}}^{\infty, \text{temp}})$ has been defined in [KS01] where it is denoted $\mathcal{O}_{X_{\text{sa}}}^t$.

Proposition 6.4. *The natural morphism $\text{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sa}}^! \mathcal{O}_{\overline{X}_{\text{sa}}}, \mathcal{C}_{X_{\text{sa}}}^{\infty, tp}) \rightarrow \mathcal{O}_{X_{\text{sa}}}^{\text{temp}}$ is an isomorphism in $D^+(\mathcal{D}_{X_{\text{sa}}})$.*

Sketch of proof. By mimicking the proof of [KS96, Th. 10.5], we are reduced to prove that, denoting by Δ the Laplace operator on $M = \mathbb{R}^n$, for any $U \in \text{Op}_{M_{\text{sa}}}$ we have the following.

$$\begin{aligned} \text{Let } \varphi \in \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{temp}}(U) \text{ and assume that } \Delta\varphi \in \mathcal{C}_{M_{\text{sal}}}^{\infty, tp}(U). \text{ Then} \\ \varphi \in \mathcal{C}_{M_{\text{sal}}}^{\infty, tp}(U). \end{aligned}$$

The proof goes as in [KS96, Prop. 10.1].

Q.E.D.

One also defines the sheaves of holomorphic functions of Gevrey type (s) or $\{s\}$ as:

$$\begin{aligned} \mathcal{G}\mathcal{O}_{X_{\text{sa}}}^{(s)} &:= \text{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sa}}^! \mathcal{O}_{\overline{X}_{\text{sa}}}, \mathcal{G}_{X_{\text{sa}}}^{(s)}) \in D^+(\mathcal{D}_{X_{\text{sa}}}), \\ \mathcal{G}\mathcal{O}_{X_{\text{sa}}}^{\{s\}} &:= \text{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sa}}^! \mathcal{O}_{\overline{X}_{\text{sa}}}, \mathcal{G}_{X_{\text{sa}}}^{\{s\}}) \in D^+(\mathcal{D}_{X_{\text{sa}}}). \end{aligned}$$

Denote by $\mathcal{C}_{M_{\text{sa}}}^{\infty,s,(p,q)}$ the sheaf of differential forms of type (p, q) with coefficients in $\mathcal{C}_{M_{\text{sa}}}^{\infty,s}$. We define $\mathcal{O}_{X_{\text{sa}}}^s$ by the Dolbeault complex

$$\mathcal{O}_{X_{\text{sa}}}^s := 0 \rightarrow \mathcal{C}_{M_{\text{sa}}}^{\infty,s,(0,0)} \xrightarrow{\bar{\partial}} \mathcal{C}_{M_{\text{sa}}}^{\infty,s+1,(0,1)} \xrightarrow{\bar{\partial}} \dots \rightarrow \mathcal{C}_{M_{\text{sa}}}^{\infty,s+d_X,(0,d_X)} \rightarrow 0.$$

One shall be aware that in general $\mathcal{C}_{M_{\text{sa}}}^{\infty,s,(p,q)}$ is not in degree 0.

Remark 6.5. We denote by $F\mathcal{D}_M$ the ring \mathcal{D}_M endowed with the natural filtration by the order and we use a similar notation for the filtered rings $F\mathcal{D}_{M_{\text{sa}}}$ and $F\mathcal{D}_{M_{\text{sal}}}$.

Recall after Schneiders [Sn99] that the category $\text{Mod}(F\mathcal{D}_M)$ is quasi-abelian and not abelian in general, but one can nevertheless define the derived categories $D(F\mathcal{D}_M)$ and $D^+(F\mathcal{D}_M)$ and similarly with $F\mathcal{D}_{M_{\text{sa}}}$ and $F\mathcal{D}_{M_{\text{sal}}}$. In a forthcoming paper, we shall prove an analogue of Theorem 6.1 in the framework of these filtered rings.

Hence, denoting by $FC_{M_{\text{sal}}}^{\infty,\text{temp}}$ the filtration on $\mathcal{C}_{M_{\text{sal}}}^{\infty,\text{temp}}$ given by Proposition 5.4, we have $FC_{M_{\text{sal}}}^{\infty,\text{temp}} \in \text{Mod}(F\mathcal{D}_{M_{\text{sal}}})$. Then we will define the filtered sheaf of holomorphic functions on X_{sa} as:

$$F\mathcal{O}_{X_{\text{sa}}}^{\text{temp}} := \text{R}\mathcal{H}om_{F\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sa}}! \mathcal{O}_{\overline{X}_{\text{sa}}}, FC_{X_{\text{sa}}}^{\infty,\text{temp}}) \in D^+(F\mathcal{D}_{X_{\text{sa}}}).$$

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